EXISTENCE AND UNIQUENESS THEOREMS FOR ORDINARY DIFFERENTIAL EQUATIONS WITH LINEAR PROGRAMS EMBEDDED *

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1. Introduction. This work discusses existence and uniqueness theorems for the solution of an initial value problem in ordinary differential equations (ODEs) with a linear program (LP) embedded. Specifically, the solution set of a LP influences the vector field of the ODE, meanwhile the LP has a parametric dependence on the differential state through the right-hand sides of its constraints. These results are fairly general and are based on known results from viability theory.

2. Problem Statement and Preliminaries. Let $D_t \subset \mathbb{R}$, $D_x \subset \mathbb{R}^{n_x}$ and $D_g \subset \mathbb{R}^{n_g}$ be open connected sets. Let $\mathcal{U}$ be a subset of $\mathcal{P}(\mathbb{R}^{n_u})$. Let $\mathbf{f} : D_t \times D_x \times D_g \to \mathbb{R}^{n_x}$, $\mathbf{g} : \mathcal{U} \to \mathbb{R}^{n_g}$, $\mathbf{b} : D_t \times D_x \to \mathbb{R}^m$, $\mathbf{A} \in \mathbb{R}^{m \times n_u}$, and $\mathbf{c} \in \mathbb{R}^{n_u}$ be given. First, let

$$\begin{aligned} q(\mathbf{d}) = \inf_{\mathbf{v} \in \mathbb{R}^{n_u}} \ & \mathbf{c}^{\mathrm{T}} \mathbf{v} \\ \text{s.t. } & \mathbf{A}\mathbf{v} = \mathbf{d}, \\ & \mathbf{v} \geq \mathbf{0}. \end{aligned}$$

Subsequently, define $F \equiv \{\mathbf{d} \in \mathbb{R}^m : -\infty < q(\mathbf{d}) < +\infty\}$. Finally, define $K \subset D_t \times D_x$ by $K \equiv \mathbf{b}^{-1}(F)$.

The focus of this work is an initial value problem in ODEs: given a $t_0 \in D_t$ and $\mathbf{x}_0 \in D_x$, we seek an interval $[t_0, t_f] = I \subset D_t$, and (absolutely continuous) function $\mathbf{x} \in C(I, D_x)$ which satisfy

$$\begin{aligned} &\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{g} \circ U(t, \mathbf{x}(t))), \ a.e.\ t \in (t_0, t_f], \\ &\mathbf{x}(t_0) = \mathbf{x}_0, \end{aligned} \tag{2.1}$$

where $U : K \rightrightarrows \mathbb{R}^{n_u}$ is given by

$$\begin{aligned} U(t, \mathbf{z}) = \arg \min_{\mathbf{v} \in \mathbb{R}^{n_u}} \ & \mathbf{c}^{\mathrm{T}} \mathbf{v} \\ \text{s.t. } & \mathbf{A}\mathbf{v} = \mathbf{b}(t, \mathbf{z}), \\ & \mathbf{v} \geq \mathbf{0}. \end{aligned} \tag{2.2}$$

Such an I and $\mathbf{x}$ will be called a *solution* of (2.1).

The following preliminaries will be helpful. It should be clear that the solution set of LP (2.2), $U(t, \mathbf{z})$, is nonempty $\forall (t, \mathbf{z}) \in K$. From sensitivity analysis and duality theory, F is either empty or $F = \{\mathbf{A}\mathbf{v} \in \mathbb{R}^m : \mathbf{v} \geq \mathbf{0}\}$ (see Table 4.2 of [2]). In either case, F is a closed set.

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Another useful definition will be $\widehat{U} : F \rightrightarrows \mathbb{R}^{n_u}$ as defined by

$$\begin{aligned} \widehat{U}(\mathbf{d}) = \arg \min_{\mathbf{v} \in \mathbb{R}^{n_u}} \ & \mathbf{c}^{\mathrm{T}} \mathbf{v} \\ \text{s.t. } & \mathbf{A}\mathbf{v} = \mathbf{d}, \\ & \mathbf{v} \geq \mathbf{0}. \end{aligned} \tag{2.3}$$

$\widehat{U}$ is the solution set as a function of the right-hand side of the equality constraints of the LP. Note that $U(t, \mathbf{z}) = \widehat{U}(\mathbf{b}(t, \mathbf{z}))$, $\forall (t, \mathbf{z}) \in K$.

The following assumptions will be fairly critical to the work that follows. Most theoretical results will require Assumptions 2.1.

ASSUMPTION 2.1. *Let* $\widehat{\mathcal{S}} = \{\widehat{U}(\mathbf{d}) : \mathbf{d} \in F\}$, $\mathbf{g} : \widehat{\mathcal{S}} \to \mathbb{R}^{n_g}$*. Assume*

(i) $\mathbf{g} \circ \widehat{U}$ *is continuous on* F,
(ii) $\mathbf{g} \circ \widehat{U}$ *is locally Lipschitz continuous on* F,
(iii) $\mathbf{g} \circ \widehat{U}$ *is Lipschitz continuous on* F.

While Assumption 2.1 (iii) implies Assumption 2.1 (ii) implies Assumption 2.1 (i), it will be helpful to be able to refer to them separately.

The contingent cone (sometimes called the Bouligand tangent cone) $T_V(\mathbf{v})$ of a set $V \subset \mathbb{R}^n$ at $\mathbf{v} \in \overline{V}$ is given by

$$T_V(\mathbf{v}) = \left\{ \mathbf{w} \in \mathbb{R}^n : \liminf_{h \to 0^+} \frac{d(\mathbf{v} + h\mathbf{w}, V)}{h} = 0 \right\}$$

where $d(\mathbf{z}, V) = \inf_{\mathbf{v} \in V} \|\mathbf{z} - \mathbf{v}\|$. A basic property is that if $\mathbf{v} \in \operatorname{int} V$, then $T_V(\mathbf{v}) = \mathbb{R}^n$. The following lemmata establish other properties.

LEMMA 2.2. *If* $\mathbf{w} \in T_V(\mathbf{v})$*, then for any open set* $N \ni \mathbf{v}$, $\mathbf{w} \in T_{V \cap N}(\mathbf{v})$.

Proof. If $\mathbf{w} \in T_V(\mathbf{v})$, an equivalent characterization is that there exist sequences $\{h_n > 0 : n \geq 1\}$, $h_n \to 0^+$ and $\{\mathbf{w}_n : n \geq 1\}$, $\mathbf{w}_n \to \mathbf{w}$ such that for all $n \geq 1$, $\mathbf{v} + h_n \mathbf{w}_n \in V$ (cf. §1.1 of [1]). Since N is open, for sufficiently small h, $\mathbf{v} + h\widehat{\mathbf{w}} \in N$ for any $\widehat{\mathbf{w}}$. Thus, there are subsequences of h_n and $\mathbf{w}_n$, $\{h_{n_k} : k \geq 1\}$ and $\{\mathbf{w}_{n_k} : k \geq 1\}$, such that $\mathbf{v} + h_{n_k} \mathbf{w}_{n_k} \in N$ for all $k \geq 1$. Consequently, $\mathbf{v} + h_{n_k} \mathbf{w}_{n_k} \in V \cap N$ for all $k \geq 1$, and so $\mathbf{v} \in \overline{V \cap N}$ and $\mathbf{w} \in T_{V \cap N}(\mathbf{v})$. ∎

LEMMA 2.3. *If* $\mathbf{w} \in T_V(\mathbf{v})$, $\mathbf{v} = (v, \widetilde{\mathbf{v}})$ *and* $\mathbf{w} = (1, \widetilde{\mathbf{w}})$*, then for any* $R = [v_a, v_b) \times \widetilde{N}$*, where* $\widetilde{N}$ *is an open set containing* $\widetilde{\mathbf{v}}$ *and* $v_a \leq v < v_b$, $\mathbf{w} \in T_{V \cap R}(\mathbf{v})$.

Proof. The proof proceeds similarly to that of Lemma 2.2. There exist sequences $h_n \to 0^+$ and $(y_n, \widetilde{\mathbf{w}}_n) \to (1, \widetilde{\mathbf{w}})$, such that $(v, \widetilde{\mathbf{v}}) + h_n(y_n, \widetilde{\mathbf{w}}_n) \in V$ for all n. Then, for large enough n, $(v, \widetilde{\mathbf{v}}) + h_n(y_n, \widetilde{\mathbf{w}}_n) \in R$ since $\widetilde{N}$ is open and v is a limit point of $[v_a, v_b)$. Thus there are subsequences such that $(v, \widetilde{\mathbf{v}}) + h_{n_k}(y_{n_k}, \widetilde{\mathbf{w}}_{n_k}) \in V \cap R$, and it follows that $\mathbf{v} \in \overline{V \cap R}$ and $\mathbf{w} \in T_{V \cap R}(\mathbf{v})$. ∎

The following concepts and results from viability theory (see Ch. 1 of [1]) will be useful in some of the theorems in the following section. First, let $\widehat{\mathbf{f}} : D_x \to \mathbb{R}^{n_x}$. Consider the initial value problem in an autonomous ODE:

$$\begin{aligned} &\dot{\mathbf{x}}(t) = \widehat{\mathbf{f}}(\mathbf{x}(t)), \ \forall t \in [t_0, t_f], \\ &\mathbf{x}(t_0) = \mathbf{x}_0. \end{aligned} \tag{2.4}$$

DEFINITION 2.4. *Let* $V \subset \mathbb{R}^{n_x}$*. A function* $\mathbf{x}$ *from* $[t_0, t_f]$ *to* $\mathbb{R}^{n_x}$ *is* viable in V on $[t_0, t_f]$ *if* $\mathbf{x}(t) \in V$ *for all* $t \in [t_0, t_f]$,

DEFINITION 2.5. *Let* $V \subset D_x$*.* V *is* locally viable under $\widehat{\mathbf{f}}$ *if for any initial state* $\mathbf{x}_0 \in V$*, there exists* $t_f > t_0$ *and a viable function* $\mathbf{x}$ *on* $[t_0, t_f]$ *such that* $[t_0, t_f]$ *and* $\mathbf{x}$ *are a solution to problem* (2.4).

DEFINITION 2.6. *Let $V \subset D_x$. V is a* viability domain *of $\widehat{\mathbf{f}}$ if*

$$\widehat{\mathbf{f}}(\mathbf{z}) \in T_V(\mathbf{z}), \forall \mathbf{z} \in V.$$

THEOREM 2.7. *Assume that V is locally compact and $\widehat{\mathbf{f}}$ is continuous from V to $\mathbb{R}^{n_x}$. Then V is locally viable under $\widehat{\mathbf{f}}$ if and only if V is a viability domain of $\widehat{\mathbf{f}}$.*

Thm. 2.7 is often called the Nagumo Theorem; for a proof see §1.2 of [1]. V is a locally compact space if every point of V has a neighborhood whose closure is compact [9]. A few basic propositions concerning locally compact metric spaces are as follows; the reader is referred to [7] for proofs and further background.

1. $\mathbb{R}^n$ is locally compact.
2. Closed subsets of locally compact metric spaces inherit local compactness.
3. Open subsets of locally compact metric spaces inherit local compactness.
4. The finite product of locally compact metric spaces is locally compact; i.e. if X and Y are locally compact metric spaces then so is $X \times Y$.
5. For $t_0 < t_1$, $[t_0, t_1) \subset \mathbb{R}$ is locally compact.

3. Existence and Uniqueness.

The complication in proving an existence result comes from the fact that K could be empty, or $\mathbf{g} \circ U(t, \mathbf{z}) \notin D_g$ for some $(t, \mathbf{z}) \in D_t \times D_x$. As well, K is not necessarily open, and since it is the domain of definition for the LP (2.2), the solution ceases to exist if $(t, \mathbf{x}(t))$ leaves K. Because of considerations such as these, in general one cannot apply an existence theorem such as Peano's directly to (2.1).

3.1. Carathéodory Systems.

The following theorem provides a solution in the sense of Carathéodory. Such a theorem is useful for applications involving controls, however there is a trade-off as its assumptions imply that the initial point $(t_0, \mathbf{x}_0)$ cannot be a boundary point of K. In many situations this is not a restrictive assumption, but it must be kept in mind.

THEOREM 3.1. *Let Assumption 2.1* (i) *hold. If*

1. *$\exists$ an open neighborhood N_0 of $(t_0, \mathbf{x}_0)$* s.t. *$\mathbf{g} \circ U(t, \mathbf{z}) \in D_g$, $\forall (t, \mathbf{z}) \in N_0$,*
2. *$\exists$ a Lebesgue-integrable function $m(t)$* s.t. *$\|\mathbf{f}(t, \mathbf{z}, \mathbf{v})\| \le m(t)$, $\forall (t, \mathbf{z}, \mathbf{v}) \in N_0 \times \mathbf{g} \circ U(N_0)$,*
3. *$\mathbf{f}(\cdot, \mathbf{z}, \mathbf{v})$ is measurable $\forall (\mathbf{z}, \mathbf{v}) \in D_x \times D_g$,*
4. *$\mathbf{f}(t, \cdot, \cdot)$ is continuous a.e. $t \in D_t$,*
5. *$\mathbf{b}(\cdot, \mathbf{z})$ is measurable $\forall \mathbf{z} \in D_x$, and*
6. *$\mathbf{b}(t, \cdot)$ is continuous a.e. $t \in D_t$,*

then a solution of Eqn. (2.1) *exists.*

Proof. Note that if $\mathbf{g} \circ U$ is defined on N_0, then $N_0 \subset K$. Moreover, N_0 contains a set $R = I_t \times I_x \ni (t_0, \mathbf{x}_0)$ where $I_t \subset D_t$, $I_x \subset D_x$, and I_t and I_x are connected sets.

Since $\mathbf{g} \circ \widehat{U}$ is continuous on F, the composition $\mathbf{g} \circ \widehat{U} \circ \mathbf{b}(t, \cdot) = \mathbf{g} \circ U(t, \cdot)$ is continuous for *a.e.* $t \in D_t$. Similarly, $\mathbf{g} \circ \widehat{U} \circ \mathbf{b}(\cdot, \mathbf{z}) = \mathbf{g} \circ U(\cdot, \mathbf{z})$ is measurable $\forall \mathbf{z} \in D_x$.

Then, by the assumptions on $\mathbf{f}$, it follows that $\mathbf{f}(t, \cdot, \mathbf{g} \circ U(t, \cdot))$ is defined and continuous on I_x for *a.e.* $t \in I_t$. Further, $\mathbf{f}(\cdot, \mathbf{z}, \mathbf{g} \circ U(\cdot, \mathbf{z}))$ is defined and measurable on I_t for all $\mathbf{z} \in I_x$. Finally, $\|\mathbf{f}(t, \mathbf{z}, \mathbf{g} \circ U(t, \mathbf{z}))\| \le m(t)$ for all $(t, \mathbf{z})$ in $I_t \times I_x$. Consequently, one can immediately apply the Carathéodory existence theorem, see for example Thm. 1 of §1 in [4]. Thus a solution exists. ∎

The following theorem establishes a uniqueness result. It follows a fairly classical approach, see for instance Thm. 2.1 in Ch. 1 of [3]. However, the possibility of discontinuities in $\mathbf{g} \circ U$ necessitates stronger (non-local) Lipschitz continuity hypotheses.

THEOREM 3.2. *Let Assumption 2.1* (iii) *hold. If the hypotheses of Theorem 3.1 are satisfied, and if in addition:*

1. $\forall (s, \mathbf{z}) \in D_t \times D_x$, *there exist open neighborhoods* $N_t^f \subset D_t$, $N_x^f \subset D_x$ *of* s, $\mathbf{z}$, *respectively, such that for a.e.* $t \in N_t^f$, $\mathbf{f}(t, \cdot, \cdot)$ *is Lipschitz continuous on* $N_x^f \times D_g$ *with some Lipschitz constant that holds for a.e.* $t \in N_t^f$,

2. $\forall (s, \mathbf{z}) \in D_t \times D_x$, *there exist open neighborhoods* $N_t^b \subset D_t$, $N_x^b \subset D_x$ *of* $s, \mathbf{z}$, *respectively, such that for a.e.* $t \in N_t^b$, $\mathbf{b}(t, \cdot)$ *is Lipschitz continuous on* N_x^b *with some Lipschitz constant that holds for a.e.* $t \in N_t^b$,

then a unique solution of Eqn. (2.1) *exists.*

Proof. By Thm. 3.1 at least one solution must exist. For a contradiction, assume two distinct solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ exist. Assume without loss of generality that $\mathbf{x}_1(t_0) = \mathbf{x}_2(t_0) = \mathbf{x}_0$ and $\mathbf{x}_1(t) \neq \mathbf{x}_2(t)$, $\forall t \in (t_0, t_f]$.

By Hypotheses 1 and 2, there exist open neighborhoods N_t, N_x of t_0, $\mathbf{x}_0$, respectively, such that for *a.e.* $t \in N_t$, $\mathbf{f}(t, \cdot, \cdot)$ is Lipschitz continuous on $N_x \times D_g$ and $\mathbf{b}(t, \cdot)$ is Lipschitz continuous on N_x. Let $\widehat{I} = [t_0, t_f] \cap N_t \cap \mathbf{x}_1^{-1}(N_x) \cap \mathbf{x}_2^{-1}(N_x)$. Note that since $\mathbf{x}_1$ and $\mathbf{x}_2$ are continuous and N_x is open, $\widehat{I} = [t_0, t_a)$, for some $t_a > t_0$. Let $\Pi_K(t) = \{\mathbf{z} \in N_x : (t, \mathbf{z}) \in K\}$. It follows that for *a.e.* $t \in \widehat{I}$ and for $i \in \{1, 2\}$, $\mathbf{x}_i(t) \in \Pi_K(t)$.

Now, since $\mathbf{g} \circ \widehat{U}$ is Lipschitz continuous on F, and $\mathbf{b}(t, \cdot)$ is Lipschitz continuous on N_x for *a.e.* $t \in \widehat{I}$, then $\mathbf{g} \circ \widehat{U}(\mathbf{b}(t, \cdot)) = \mathbf{g} \circ U(t, \cdot)$ is Lipschitz continuous on $\Pi_K(t)$ for *a.e.* $t \in \widehat{I}$ (with some Lipschitz constant that holds for *a.e.* $t \in \widehat{I}$). Finally, $\mathbf{f}(t, \cdot, \cdot)$ is Lipschitz continuous on $N_x \times D_g$ for *a.e.* $t \in \widehat{I}$, and so $\mathbf{f}(t, \cdot, \mathbf{g} \circ U(t, \cdot))$ is Lipschitz continuous on $\Pi_K(t)$.

Let the Lipschitz constant for $\mathbf{f}(t, \cdot, \mathbf{g} \circ U(t, \cdot))$ be k, which holds for *a.e.* $t \in \widehat{I}$. Thus, for *a.e.* $t \in \widehat{I}$,

$$\|\mathbf{f}(t, \mathbf{x}_1(t), \mathbf{g} \circ U(t, \mathbf{x}_1(t))) - \mathbf{f}(t, \mathbf{x}_2(t), \mathbf{g} \circ U(t, \mathbf{x}_2(t)))\| \leq k \|\mathbf{x}_1(t) - \mathbf{x}_2(t)\|.$$

Next, since $\mathbf{x}_i$ satisfies (2.1), $\dot{\mathbf{x}}_i(s) = \mathbf{f}(s, \mathbf{x}_i(s), \mathbf{g} \circ U(s, \mathbf{x}_i(s)))$ for *a.e.* $s \in \widehat{I}$ and for $i \in \{1, 2\}$. So, one has

$$\|\dot{\mathbf{x}}_1(s) - \dot{\mathbf{x}}_2(s)\| = \|\mathbf{f}(s, \mathbf{x}_1(s), \mathbf{g} \circ U(s, \mathbf{x}_1(s))) - \mathbf{f}(s, \mathbf{x}_2(s), \mathbf{g} \circ U(s, \mathbf{x}_2(s)))\|, \ a.e.\ s \in \widehat{I},$$

and then by the above discussion,

$$\|\dot{\mathbf{x}}_1(s) - \dot{\mathbf{x}}_2(s)\| \leq k \|\mathbf{x}_1(s) - \mathbf{x}_2(s)\|, \quad a.e.\ s \in \widehat{I}. \tag{3.1}$$

Integrating (3.1) from t_0 to any t, $t \leq t_a$, yields

$$\left\| \int_{t_0}^{t} \dot{\mathbf{x}}_1(s) - \dot{\mathbf{x}}_2(s) ds \right\| \leq k \int_{t_0}^{t} \|\mathbf{x}_1(s) - \mathbf{x}_2(s)\| ds.$$

Then

$$\begin{aligned} &\|\mathbf{x}_1(t) - \mathbf{x}_2(t)\| - \|\mathbf{x}_1(t_0) - \mathbf{x}_2(t_0)\| \leq \\ &\qquad \|(\mathbf{x}_1(t) - \mathbf{x}_2(t)) - (\mathbf{x}_1(t_0) - \mathbf{x}_2(t_0))\| \leq \\ &\qquad\qquad k \int_{t_0}^{t} \|\mathbf{x}_1(s) - \mathbf{x}_2(s)\| ds. \end{aligned}$$

Finally, an application of Gronwall's inequality (see for instance Thm. 1.1 in Ch. III of [6]) yields

$$\|\mathbf{x}_1(t) - \mathbf{x}_2(t)\| \leq \|\mathbf{x}_1(t_0) - \mathbf{x}_2(t_0)\| e^{k(t - t_0)}.$$

However, $\|\mathbf{x}_1(t_0) - \mathbf{x}_2(t_0)\| = 0$, so one sees that $\|\mathbf{x}_1(t) - \mathbf{x}_2(t)\| \leq 0$, or that $\mathbf{x}_1(t) = \mathbf{x}_2(t)$ for all $t \in \widehat{I}$ which is a contradiction. Thus a unique solution exists. ∎

With the uniqueness result, one can continue the solution for as long as the existence results hold; however one must be aware that the additional hypotheses, described below, are required once $(t, \mathbf{x}(t))$ becomes a boundary point of K.

3.2. General Initial Point. The general case – i.e. any initial point – is now considered. This case relies on the Nagumo Theorem. As previously stated, this theorem furnishes a continuously differentiable solution $\mathbf{x}$, rather than an absolutely continuous one (as would be obtained from a Carathéodory equation). Extensions of the Nagumo theorem to Carathéodory-type equations have been explored in the literature; most stem from set-valued analysis and solutions for differential inclusions. For instance [5] gives a necessary and sufficient condition for existence of a solution of a differential inclusion with time varying constraints, which is very similar to what occurs with problem (2.1). However, the assumptions required on the set K are too restrictive for the present purposes.

THEOREM 3.3. *Let Assumption 2.1* (i) *hold. If*

1. $(t_0, \mathbf{x}_0) \in K$,
2. $\mathbf{g} \circ U(t_0, \mathbf{x}_0) \in D_g$,
3. $\mathbf{f}$ *is continuous,*
4. $\mathbf{b}$ *is continuous, and*
5. *there exist* $t_1 \in D_t$, $t_1 > t_0$, *and an open set* $N_x \subset D_x$ *containing* $\mathbf{x}_0$ *such that* $(1, \mathbf{f}(t, \mathbf{z}, \mathbf{g} \circ U(t, \mathbf{z}))) \in T_K(t, \mathbf{z})$ *for all* $(t, \mathbf{z}) \in K \cap N_0$ *where* $N_0 \equiv [t_0, t_1) \times N_x$,

then a solution of Eqn. (2.1) *exists.*

Proof. We will construct a set that is a viability domain for an equivalent autonomous system and apply Thm. 2.7.

First, $\mathbf{g} \circ \widehat{U}$ is continuous on F, so $\mathbf{g} \circ U$ is continuous on K, and so $(\mathbf{g} \circ U)^{-1}(D_g)$ is open in K. By simple topological arguments, this means that $(\mathbf{g} \circ U)^{-1}(D_g) = K \cap N_1$ for some open $N_1 \subset \mathbb{R}^{1+n_x}$, and we can further assume that $N_1 \subset D_t \times D_x$. Note that $(t_0, \mathbf{x}_0) \in (\mathbf{g} \circ U)^{-1}(D_g) \subset N_1$.

Let $\widehat{K} = K \cap N_1 \cap N_0$. Note that $\widehat{K}$ is nonempty, since $(t_0, \mathbf{x}_0)$ is in each of K, N_1, N_0. More importantly, it is locally compact. To see this, first note that $[t_0, t_1)$ and N_x are locally compact, and so N_0 is also locally compact. Then, since $N_1 \cap N_0$ is an open subset of N_0, it too is locally compact. Finally, $\widehat{K} = K \cap N_1 \cap N_0 = \mathbf{b}^{-1}(F)$ is a closed subset of $N_1 \cap N_0$, since $\mathbf{b}$ is continuous on $N_1 \cap N_0$ and F is closed. As a closed subset of a locally compact space, $\widehat{K}$ is locally compact.

Note that $\mathbf{g} \circ U$ is defined, continuous, and takes values in D_g on $\widehat{K}$, and so $\mathbf{f}(\cdot, \cdot, \mathbf{g} \circ U(\cdot, \cdot))$ is defined and continuous on $\widehat{K}$.

Now, let

$$\widehat{\mathbf{f}} : \widehat{K} \to \mathbb{R}^{1+n_x} : (t, \mathbf{z}) \mapsto (1, \mathbf{f}(t, \mathbf{z}, \mathbf{g} \circ U(t, \mathbf{z}))).$$

By construction, $\widehat{\mathbf{f}}$ is continuous on $\widehat{K}$. Introduce the dummy variable s and formulate the initial value problem

$$\dot{\mathbf{y}}(s) = \widehat{\mathbf{f}}(\mathbf{y}(s)), \quad \mathbf{y}(s_0) = (t_0, \mathbf{x}_0). \tag{3.2}$$

The value of s is immaterial, so let $s_0 = t_0$. If there exists a solution $\mathbf{y}(s) = (t(s), \mathbf{x}(s))$ of (3.2), then it follows that $\frac{dt}{ds}(s) = 1$ and that $t(s) = s$. Furthermore,

$$\frac{d\mathbf{x}}{ds}(s) = \mathbf{f}(t(s), \mathbf{x}(s), \mathbf{g} \circ U(t(s), \mathbf{x}(s))) = \mathbf{f}(s, \mathbf{x}(s), \mathbf{g} \circ U(s, \mathbf{x}(s))).$$

Thus there would exist a corresponding solution of (2.1).

To show that $\widehat{K}$ is a viability domain of $\widehat{\mathbf{f}}$, pick any $(t, \mathbf{z}) \in \widehat{K}$. Then $(t, \mathbf{z}) \in K \cap N_0$, and by assumption $\widehat{\mathbf{f}}(t, \mathbf{z}) \in T_K(t, \mathbf{z})$. So by Lemma 2.2, $\widehat{\mathbf{f}}(t, \mathbf{z}) \in T_{K \cap N_1}(t, \mathbf{z})$. Then, by Lemma 2.3, $\widehat{\mathbf{f}}(t, \mathbf{z}) \in T_{K \cap N_1 \cap N_0}(t, \mathbf{z})$.

Thus, for all $(t, \mathbf{z}) \in \widehat{K}$, $\widehat{\mathbf{f}}(t, \mathbf{z}) \in T_{\widehat{K}}(t, \mathbf{z})$, and so $\widehat{K}$ is a viability domain of $\widehat{\mathbf{f}}$, and by Thm. 2.7 a solution exists for (3.2), which corresponds to a solution of (2.1). ∎

Finally, a uniqueness result for general initial points follows. Since $\mathbf{b}$ follows stronger continuity properties, the Lipschitz continuity requirements can be relaxed to local Lipschitz continuity.

THEOREM 3.4. *Let Assumption 2.1* (ii) *hold. If the hypotheses of Thm. 3.3 hold, and if in addition*

1. *$\forall (s, \mathbf{z}, \mathbf{v}) \in D_t \times D_x \times D_g$, there exist open neighborhoods $N_t^f \subset D_t$, $N_x^f \subset D_x$, $N_g \subset D_g$ of s, $\mathbf{z}$, $\mathbf{v}$, respectively, such that for a.e. $t \in N_t^f$, $\mathbf{f}(t, \cdot, \cdot)$ is Lipschitz continuous on $N_x^f \times N_g$ with some Lipschitz constant that holds for a.e. $t \in N_t^f$,*
2. *$\forall (s, \mathbf{z}) \in D_t \times D_x$, there exist open neighborhoods $N_t^b \subset D_t$, $N_x^b \subset D_x$ of $s, \mathbf{z}$, respectively, such that for a.e. $t \in N_t^b$, $\mathbf{b}(t, \cdot)$ is Lipschitz continuous on N_x^b with some Lipschitz constant that holds for a.e. $t \in N_t^b$,*

then a unique solution of (2.1) *exists.*

Proof. By Thm. 3.3, at least one solution must exist. For a contradiction, assume two distinct solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ exist. Assume without loss of generality that $\mathbf{x}_1(t_0) = \mathbf{x}_2(t_0) = \mathbf{x}_0$ and $\mathbf{x}_1(t) \neq \mathbf{x}_2(t)$, $\forall t \in (t_0, t_f]$.

Let N_g be the open neighborhood of $\mathbf{g} \circ U(t_0, \mathbf{x}_0)$ as in Hypothesis 1. Let N_b be the open neighborhood of $\mathbf{b}(t_0, \mathbf{x}_0)$ on which $\mathbf{g} \circ \widehat{U}$ is Lipschitz continuous. Let $N_t = N_t^f \cap N_t^b$ and $N_x = N_x^f \cap N_x^b$ be the open neighborhoods of t_0, $\mathbf{x}_0$, respectively, such that for *a.e.* $t \in N_t$, $\mathbf{f}(t, \cdot, \cdot)$ is Lipschitz continuous on $N_x \times N_g$ and $\mathbf{b}(t, \cdot)$ is Lipschitz continuous on N_x.

Since $\mathbf{x}_i$ $(i \in \{1, 2\})$, $\mathbf{b}$ and $\mathbf{g} \circ \widehat{U}$ are continuous, so are $\mathbf{b}(\cdot, \mathbf{x}_i(\cdot))$ and $\mathbf{g} \circ U(\cdot, \mathbf{x}_i(\cdot))$ $(i \in \{1, 2\})$. Note that the neighborhood N_b is actually open *with respect to F*. This is not an issue, though, because the functions $\mathbf{b}(\cdot, \mathbf{x}_i(\cdot))$ are still continuous functions from $[t_0, t_f]$ into F, and so the preimage of N_b under $\mathbf{b}(\cdot, \mathbf{x}_i(\cdot))$ is open in $[t_0, t_f]$. Thus

$$\widetilde{I} \equiv N_t \cap \left(\bigcap_{i \in \{1,2\}} \left(\mathbf{x}_i^{-1}(N_x) \cap (\mathbf{b}(\cdot, \mathbf{x}_i(\cdot)))^{-1}(N_b) \cap (\mathbf{g} \circ U(\cdot, \mathbf{x}_i(\cdot)))^{-1}(N_g) \right) \right)$$

is open in $[t_0, t_f]$ and contains t_0. Let $\widehat{I}$ be a subset of $\widetilde{I}$ of the form $[t_0, t_a)$, $t_a > t_0$.

It follows that for *a.e.* $t \in \widehat{I}$,

$$\|\mathbf{f}(t, \mathbf{x}_1(t), \mathbf{g} \circ U(t, \mathbf{x}_1(t))) - \mathbf{f}(t, \mathbf{x}_2(t), \mathbf{g} \circ U(t, \mathbf{x}_2(t)))\| \leq k \|\mathbf{x}_1(t) - \mathbf{x}_2(t)\|$$

for some positive, finite k. At this point, the proof proceeds exactly as that of Thm. 3.2. We reach the contradiction that $\|\mathbf{x}_1(t) - \mathbf{x}_2(t)\| \leq 0$, or that $\mathbf{x}_1(t) = \mathbf{x}_2(t)$ for all $t \in \widehat{I}$, and so a unique solution exists. ∎

The combination of all these existence and uniqueness results provide conditions that guarantee that the system (2.1) and (2.2) are amenable to numerical integration.

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