# Well-Posedness Results for Carathéodory Index-1 Semi-Explicit Differential-Algebraic Equations

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**Abstract**

The well-posedness of nonsmooth differential-algebraic equations (DAEs) is investigated. More specifically, semi-explicit DAEs with Carathéodory-style assumptions on the differential right-hand side functions and local Lipschitz continuity assumptions on the algebraic equations. The DAEs are classified as having differential index one in a generalized sense; solution regularity is formulated in terms of projections of generalized (Clarke) Jacobians. Consistent initialization is resolved via Clarke's nonsmooth implicit function theorem. Existence of solutions is derived under consistency and regularity of the initial data. Uniqueness of a solution is guaranteed under analogous Carathéodory ODE uniqueness assumptions. The continuation of such solutions is established and sufficient conditions for continuous and Lipschitzian parametric dependence of solutions are also provided. To accomplish these results, a theoretical tool for analyzing nonsmooth DAEs is provided in the form of an extended nonsmooth implicit function theorem. The findings here are a natural extension of classical results and lay the foundation for further theoretical and computational analyses of nonsmooth DAEs.

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## 1. Introduction

Differential-algebraic equations (DAEs) provide a natural framework for the dynamic modeling and simulation of a wide range of engineering applications found in network modeling, mechanical multibody systems, constrained variational problems, and fluid dynamics (see [1, 2] and the references therein). Nonsmoothness is an inherent feature of dynamic models of chemical processes [3]. For example, sources of nonsmoothness in campaign continuous pharmaceutical manufacturing include thermodynamic phase changes (e.g., flash evaporation, liquid-liquid extraction), flow transitions (e.g., laminar-turbulent-choked transitions), flow control devices (e.g., nonreturn valves, weirs), crystallization kinetics that vary whether the solution is supersaturated or unsaturated, etc. [4–6]. Flash processes in which a feed can be separated into liquid and vapor phases using pressure and/or heat can be dynamically modeled using a nonsmooth reformulation [7].

Consider the closed rigid vessel being heated as depicted in Figure 1, modeled as the following dynamical system:

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$$\begin{aligned}
\dot{U}(t) &= \dot{Q}(t), &\text{(1a)}\\
V_T &= V_L(t) + V_V(t), &\text{(1b)}\\
M_i &= M_L(t)x_i(t) + M_V(t)y_i(t), \quad i = 1, 2, &\text{(1c)}\\
y_i(t) &= K_i(T(t), P(t))x_i(t), \quad i = 1, 2, &\text{(1d)}\\
\sum_{i=1}^{2} M_i &= M_L(t) + M_V(t), &\text{(1e)}\\
0 &= \operatorname{mid}\left(\frac{M_V(t)}{\sum_{i=1}^{2} M_i}, \sum_{i=1}^{2} x_i(t) - \sum_{i=1}^{2} y_i(t), \frac{M_V(t)}{\sum_{i=1}^{2} M_i} - 1\right), &\text{(1f)}
\end{aligned}$$

where a number of algebraic equations (e.g., equations of state and other thermodynamic relations) have been omitted for brevity. Here $x_i(t)$, $y_i(t)$ correspond to the mole fractions of the $i^{\text{th}}$ species in the liquid phase and vapor phase at time $t$, respectively. The equilibrium ratios $K_i$ appearing in (1d) are the ratio of the mole fraction of species $i$ in vapor phase versus liquid phase, and depend only on the system's temperature, $T(t)$, and pressure, $P(t)$, under Raoult's law.

By conservation of energy, the internal energy of the system, $U(t)$, changes in time from an external heating source according to (1a). The heat duty, $\dot{Q}(t)$, satisfies

$$\dot{Q}(t) = hA(T_{\text{out}} - T(t)),$$

where $h > 0$ is the overall heat transfer coefficient, $A > 0$ is the total area for heat transfer, and $T_{\text{out}}$ is the ambient temperature outside the vessel. Since there are no material flows into or out of the closed system, the total hold-up $M_i$ of species $i$ is constant in time. Equation (1e) represents total mass balance between the species hold-ups and the constituent liquid and vapor phase hold-ups, $M_L(t)$ and $M_V(t)$, respectively. Since the vessel is rigid, the (constant) total volume $V_T$ is equal to the sum of the liquid volume $V_L(t)$ and the vapor volume $V_V(t)$ (Equation (1b)).

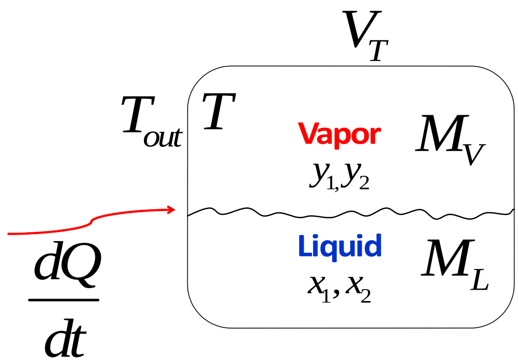



Figure 1: A closed rigid vessel with an external heat source and two species distributed between liquid and vapor phases.

The source of nonsmoothness in this DAE model is found in (1f); the mid function selects the median of its three arguments and is not differentiable everywhere. Equation (1f) enforces different algebraic constraints according to the thermodynamic phase regime in the system as follows: if there is only a liquid phase present at time $t$ then $M_V(t) = 0$. In this case, the first argument in the mid function is zero and the third argument is equal to minus one. Furthermore, it can be shown via a Gibbs free energy minimization [7] that the nonphysical vapor mole fractions $y_i$ need not sum to one but must satisfy $\sum_{i=1}^{2} x_i(t) \geq \sum_{i=1}^{2} y_i(t)$. Hence, the middle argument is nonnegative and (1f) enforces the algebraic constraint $M_V(t) = 0$ (i.e., the liquid-only phase regime). If there is no liquid phase present at time $t$ then $M_L(t) = 0$ so that $M_V(t)/(M_L(t) + M_V(t)) = 1$ and, by similar arguments, the mid function selects the third argument in this case (i.e., (1f) enforces $M_L(t) = 0$ representing the vapor-only phase regime). Lastly, if both phases are present, then $\sum_{i=1}^{2} x_i(t) = \sum_{i=1}^{2} y_i(t)$ so that the second argument in the mid function evaluates to zero. Since $0 < M_V(t), M_L(t) < (M_L(t) + M_V(t))$ in the two-phase regime, the first and third arguments are positive and negative, respectively, and the mid function selects the second argument. In summary, (1) is a nonsmooth DAE model of the rigid vessel which models dynamic transitions between the three phase regimes (i.e., vapor-only, liquid-only, and two-phase).

In analyzing such nonsmooth dynamical systems, information obtained through sensitivity analysis is valuable for nonsmooth equation-solving techniques (e.g., semismooth Newton methods [8, 9] and LP-Newton methods [10]) and optimization problems (e.g., bundle methods for local optimization [11–13]). Until recent theoretical advancements in tractable algorithms using lexicographic differentiation [14] to calculate lexicographic directional-derivatives [15], theoretical and computational approaches in nonsmooth analysis (see, e.g., [16–18] for an overview of this area of research) were lacking because of inherent difficulties in acquiring elements of some class of generalized derivative. These recent theoretical advancements give computationally relevant generalized derivatives for parametric ordinary-differential equations (ODEs) with nonsmooth right-hand sides [19], nonsmooth optimal control problems [20], hybrid systems [21], and ODEs with linear programs embedded [22]. The extension of the aforementioned theoretical and numerical generalized derivative treatments to nonsmooth DAEs requires a rigorous analysis of their well-posedness. This serves as motivation for the present article.

There have been a number of studies related to the well-posedness of solutions of smooth DAE systems (see, e.g., [23–29]). The literature on well-posedness of nonsmooth DAE systems is less extensive: DAE systems which experience discontinuities have been studied (see, e.g., [30, 31]), linear DAEs with generalized inputs and discontinuous solutions of semilinear DAEs have been investigated [32], and the solvability of nonsmooth semistate equations describing the dynamics of a circuit has been analyzed [33]. The methodologies and findings laid out in these works are unamenable to the presently motivating dynamic modeling and simulation problems because of incompatible smoothness assumptions, restrictive specializations (e.g., quasilinear form DAE systems, hybrid systems with fixed mode sequences), or undesirable abstractions (i.e., generalized solutions which unnecessarily invalidate the generalized derivative approach here).

Complementarity systems are special instances of nonsmooth dynamic systems for which treatments of systematic-theoretic issues have been made [34–37]. Complementarity systems can be equivalently formulated as nonsmooth semi-explicit DAEs using any suitable nonlinear complementarity problem (NCP) function [38]. Pang and Stewart [39, 40] introduced and investigated differential variational inequalities (DVIs), which unify a number of classes of problems (including complementarity systems). DVIs can be expressed as a class of nonsmooth DAEs by casting the variational condition as nonsmooth equations (via the natural or normap maps [38]). The well-posedness results established in the highlighted works rely on particular structures of the problems. Pang and Stewart [39] remarked that a recasting of DVIs as nonsmooth DAEs invalidates the established methodology of DAEs, except under restrictive assumptions, and therefore is only of conceptual significance. However, this is no longer the case thanks to the present article along with recent progress in nonsmooth analysis of computationally relevant generalized derivatives (as detailed earlier). Nonsmooth semi-explicit DAEs of a general form (e.g., locally Lipschitz continuous algebraic equations) are analyzed here in the spirit of the classical theory concerning well-posedness of Carathéodory ordinary-differential equations (ODEs).

A number of nonsmooth extensions of the classical implicit function theorem have been developed (see, e.g., [16, 18, 21, 41–44]) but are local in nature. A semilocal implicit function theorem was provided by Neumaier (see Theorem 5.5.1 and Corollary 5.1.5 in [45]) and has been used to compute bounds on reachable sets of semi-explicit DAEs [29]. Nonetheless, a corresponding regularity assumption on the solution trajectory is too restrictive here. An extended implicit function theorem found by Graves (see Chapter VIII, Section 4 in [46]) has been used in the DAE literature (e.g., in studying power systems [47]) but requires prohibitive smoothness assumptions. The approach here is to derive an extended nonsmooth implicit function theorem for locally Lipschitz continuous functions. In doing so, some technical results regarding projections of Clarke Jacobians are proved which make it possible to reformulate Carathéodory semi-explicit DAEs as equivalent Carathéodory ODEs on open and connected sets. Present contributions include extensions of local existence, uniqueness, and continuous dependence results to Carathéodory semi-explicit DAEs under regularity assumptions involving projections of Clarke Jacobians. Consistent initialization of such nonsmooth DAE systems is also resolved here. The techniques provided in the present manuscript may also find use in extensions to other Carathéodory-type DAEs. The findings here permit the calculation of computationally relevant generalized derivatives of nonsmooth DAE systems, as accomplished in [48], for use in nonsmooth analysis and optimization where classical methods fail.

The rest of this article is structured as follows. Necessary background is presented in Section 2: preliminaries are outlined in Section 2.1 and pertinent theory regarding generalized derivatives is given in Section 2.2. In Section 3, technical results are derived, culminating in an extended nonsmooth implicit function theorem. The nonsmooth semi-explicit DAE system of interest is formulated in Section 4: its consistent initialization is investigated in Section 4.1; results on existence and uniqueness of solutions are proved in Section 4.2; extended existence is studied in Section 4.3; and dependence of solutions on parameters is analyzed in Section 4.4. Conclusions and future directions are given in Section 5.

## 2. Mathematical Background

This section presents necessary results from nonsmooth analysis.

### *2.1. Notation and Preliminaries*

The following notational conventions are used. $\mathbb{N}$, $\mathbb{R}_+$, $\mathbb{R}^n$, and $\mathbb{R}^{m\times n}$ denote the set of positive integers, the set of nonnegative real numbers, the Euclidean space of $n$-dimensions (equipped with the Euclidean norm $\|\cdot\|$) and the vector space of $m\times n$ matrices with real-valued entries (equipped with the corresponding induced norm), respectively. Unless otherwise stated, sets are denoted by uppercase letters (e.g., $H$), matrices in $\mathbb{R}^{m\times n}$ are denoted by uppercase boldface letters (e.g., $\mathbf{H}$), elements of $\mathbb{R}$ and scalar-valued functions are denoted by lowercase letters (e.g., $h$), and vectors in $\mathbb{R}^n$ and vector-valued functions are denoted by lowercase boldface letters (e.g., $\mathbf{h}$). The zero vector in $\mathbb{R}^n$ is denoted by $\mathbf{0}_n$, the $m\times n$ zero matrix is denoted by $\mathbf{0}_{m\times n}$, and the $n\times n$ identity matrix is denoted by $\mathbf{I}_n$. A well-defined vertical block matrix (or vector):

$$\begin{bmatrix}\mathbf{H}_1\\ \mathbf{H}_2\end{bmatrix}$$

can be written as $(\mathbf{H}_1,\mathbf{H}_2)$. The $i^{\text{th}}$ component of a vector $\mathbf{h}$ is denoted by $h_i$. Parenthetical subscripts may be used to indicate the column vector of a matrix (e.g., the matrix $\mathbf{H}$ has the $k^{\text{th}}$ column $\mathbf{h}_{(k)}$, whose $i^{\text{th}}$ component is $h_{(k),i}$), or to indicate a sequence of vectors or vector-valued functions.

The *open ball of radius* $r>0$ *centered at* $\mathbf{h}\in\mathbb{R}^n$ is given by $B_r(\mathbf{h}) := \{\boldsymbol{\eta}\in\mathbb{R}^n : \|\boldsymbol{\eta}-\mathbf{h}\| < r\}$. A *neighborhood of* $\mathbf{h}\in\mathbb{R}^n$ is a set of points $B_\delta(\mathbf{h})$ for some $\delta>0$. Given a set $H\subset\mathbb{R}^n$, its *closure* and *convex hull* are denoted by $\bar{H}$ and $\operatorname{conv} H$, respectively. A *neighborhood of* $H$ is a set of points $B_\delta(H) := \cup_{\mathbf{h}\in H} B_\delta(\mathbf{h})$ for some $\delta>0$. A set of matrices in $\mathbb{R}^{n\times n}$ is said to be of *maximal rank* if it contains no singular matrices.

**Definition 2.1.** Let $n_x,n_y,n_z\in\mathbb{N}$ and $W\subset\mathbb{R}^{n_x}\times\mathbb{R}^{n_y}\times\mathbb{R}^{n_z}$. The *projections of* $W$ *onto* $\mathbb{R}^{n_x}$ *and* $\mathbb{R}^{n_x}\times\mathbb{R}^{n_y}$ are given by, respectively,

$$\pi_x W := \{\boldsymbol{\eta}_x\in\mathbb{R}^{n_x} : \exists(\boldsymbol{\eta}_x,\boldsymbol{\eta}_y,\boldsymbol{\eta}_z)\in W\}\subset\mathbb{R}^{n_x},$$
$$\pi_{x,y} W := \{(\boldsymbol{\eta}_x,\boldsymbol{\eta}_y)\in\mathbb{R}^{n_x}\times\mathbb{R}^{n_y} : \exists(\boldsymbol{\eta}_x,\boldsymbol{\eta}_y,\boldsymbol{\eta}_z)\in W\}\subset\mathbb{R}^{n_x}\times\mathbb{R}^{n_y}.$$

The projections $\pi_y W$, $\pi_z W$, $\pi_{y,z} W$, $\pi_{x,z} W$ are defined similarly. The *shadows of* $W$ *at* $\mathbf{y}\in\pi_y W$ *with respect to* $\mathbb{R}^{n_x}$ *and* $\mathbb{R}^{n_x}\times\mathbb{R}^{n_z}$ are given by, respectively,

$$\pi_x(W;\mathbf{y}) := \pi_x\{(\boldsymbol{\eta}_x,\boldsymbol{\eta}_y,\boldsymbol{\eta}_z)\in W : \boldsymbol{\eta}_y=\mathbf{y}\}\subset\mathbb{R}^{n_x},$$
$$\pi_{x,z}(W;\mathbf{y}) := \pi_{x,z}\{(\boldsymbol{\eta}_x,\boldsymbol{\eta}_y,\boldsymbol{\eta}_z)\in W : \boldsymbol{\eta}_y=\mathbf{y}\}\subset\mathbb{R}^{n_x}\times\mathbb{R}^{n_z}.$$

The *shadow of* $W$ *at* $(\mathbf{x},\mathbf{y})\in\pi_{x,y}W$ *with respect to* $\mathbb{R}^{n_z}$ is given by

$$\pi_z(W;(\mathbf{x},\mathbf{y})) := \pi_z\{(\boldsymbol{\eta}_x,\boldsymbol{\eta}_y,\boldsymbol{\eta}_z)\in W : (\boldsymbol{\eta}_x,\boldsymbol{\eta}_y)=(\mathbf{x},\mathbf{y})\}\subset\mathbb{R}^{n_z}.$$

The other non-vacuous shadows are defined similarly.

**Lemma 2.2.** Given $n_x,n_y\in\mathbb{N}$ and $W\subset\mathbb{R}^{n_x}\times\mathbb{R}^{n_y}$ open, $\pi_x W$ and $\pi_y W$ are open. Given $\mathbf{x}\in\pi_x W$ and $\mathbf{y}\in\pi_y W$, $\pi_x(W;\mathbf{y})$ and $\pi_y(W;\mathbf{x})$ are open.

*Proof.* Choose any $\mathbf{x}^*\in\pi_x W$. Choose any $\mathbf{y}^*\in\pi_y(W;\mathbf{x}^*)$, which is nonempty since $\mathbf{x}^*\in\pi_x W$. Since $W$ is open, there exists $\delta>0$ such that $B_\delta(\mathbf{x}^*,\mathbf{y}^*)\subset W$. It follows that $\pi_x B_\delta(\mathbf{x}^*,\mathbf{y}^*)\subset\pi_x W$. Choose any $\widetilde{\mathbf{x}}\in B_\delta(\mathbf{x}^*)$. Then $(\widetilde{\mathbf{x}},\mathbf{y}^*)\in B_\delta(\mathbf{x}^*,\mathbf{y}^*)$ since

$$\|(\widetilde{\mathbf{x}},\mathbf{y}^*)-(\mathbf{x}^*,\mathbf{y}^*)\| = \|\widetilde{\mathbf{x}}-\mathbf{x}^*\| < \delta.$$

Hence, $\widetilde{\mathbf{x}}\in\pi_x B_\delta(\mathbf{x}^*,\mathbf{y}^*)$ by definition. It follows that $B_\delta(\mathbf{x}^*)\subset\pi_x B_\delta(\mathbf{x}^*,\mathbf{y}^*)\subset\pi_x W$. Therefore, there exists a neighborhood of $\mathbf{x}^*$ in $\pi_x W$. $\pi_x(W;\mathbf{y})$ can be shown to be open in a similar fashion: choose any $\widehat{\mathbf{x}}$. Then $\mathbf{y}\in\pi_y(W;\widehat{\mathbf{x}})$ and, by openness of $W$, there exists $\rho>0$ such that $B_\rho(\widehat{\mathbf{x}},\mathbf{y})\subset W$. Therefore,

$$\pi_x(B_\rho(\widehat{\mathbf{x}},\mathbf{y});\mathbf{y}) = \pi_x\{(\boldsymbol{\eta}_x,\boldsymbol{\eta}_y)\in B_\rho(\widehat{\mathbf{x}},\mathbf{y}) : \boldsymbol{\eta}_y=\mathbf{y}\}\subset\pi_x\{(\boldsymbol{\eta}_x,\boldsymbol{\eta}_y)\in W : \boldsymbol{\eta}_y=\mathbf{y}\} = \pi_x(W;\mathbf{y}).$$

Choose any $\mathbf{x}^\dagger\in B_\rho(\widehat{\mathbf{x}})$. Then

$$\|(\widehat{\mathbf{x}},\mathbf{y})-(\mathbf{x}^\dagger,\mathbf{y})\| = \|\widehat{\mathbf{x}}-\mathbf{x}^\dagger\| < \rho,$$

from which it follows that $(\mathbf{x}^\dagger,\mathbf{y})\in B_\rho(\widehat{\mathbf{x}},\mathbf{y})$. Thus, $\mathbf{x}^\dagger\in\pi_x(B_\rho(\widehat{\mathbf{x}},\mathbf{y});\mathbf{y})\subset\pi_x(W;\mathbf{y})$. Consequently, $B_\rho(\widehat{\mathbf{x}})\subset\pi_x(B_\rho(\widehat{\mathbf{x}},\mathbf{y});\mathbf{y})\subset\pi_x(W;\mathbf{y})$. The proofs for $\pi_y W$ and $\pi_y(W;\mathbf{x})$ are similar. □

**Definition 2.3.** Let $n_x, n_y, n_z \in \mathbb{N}$, $W \subset \mathbb{R}^{n_x} \times \mathbb{R}^{n_y}$, $X \subset \pi_x W$, $Y \subset \pi_y W$, $(\mathbf{x}, \mathbf{y}) \in W$, and $\mathbf{f} : W \to \mathbb{R}^{n_z}$. The *cross-section of* $\mathbf{f}$ *at* $\mathbf{x}$ *and at* $\mathbf{y}$ are given by, respectively,

$$\mathbf{f}_{\mathbf{x}} : \pi_y(W; \mathbf{x}) \to \mathbb{R}^{n_z} : \boldsymbol{\eta}_y \mapsto \mathbf{f}(\mathbf{x}, \boldsymbol{\eta}_y),$$
$$\mathbf{f}_{\mathbf{y}} : \pi_x(W; \mathbf{y}) \to \mathbb{R}^{n_z} : \boldsymbol{\eta}_x \mapsto \mathbf{f}(\boldsymbol{\eta}_x, \mathbf{y}).$$

**Definition 2.4.** [49] Let $T \subset \mathbb{R}$ be connected and $\mathbf{f} : T \to \mathbb{R}^n$. $\mathbf{f}$ is said to be *absolutely continuous on* $T$ if for every compact subinterval $\bar{T} \subset T$ and every $\epsilon > 0$, there exists $\delta > 0$ such that whenever a finite sequence of pairwise disjoint subintervals $\{[a_k, b_k] : k = 1, \ldots, q, q \in \mathbb{N}\}$ of $\bar{T}$ satisfies $\sum_{k=1}^{q}(b_k - a_k) < \delta$ then $\sum_{k=1}^{q} \|\mathbf{f}(b_k) - \mathbf{f}(a_k)\| < \epsilon$.

**Lemma 2.5.** Let $T \subset \mathbb{R}$ be connected and $X \subset \mathbb{R}^n$. Let $\mathbf{h} : T \to X$ be absolutely continuous on $T$ and $\mathbf{g} : X \to \mathbb{R}^m$ be Lipschitz continuous on $X$. Then the mapping $\mathbf{f} \equiv \mathbf{g} \circ \mathbf{h} : T \to \mathbb{R}^m$ is absolutely continuous on $T$.

*Proof.* By Lipschitz continuity of $\mathbf{g}$, there exists $L_{\mathbf{g}} > 0$ such that $\|\mathbf{g}(\mathbf{x}_2) - \mathbf{g}(\mathbf{x}_1)\| \leq L_{\mathbf{g}}\|\mathbf{x}_2 - \mathbf{x}_1\|$ whenever $\mathbf{x}_1, \mathbf{x}_2 \in X$. Choose any $\epsilon > 0$ and any compact subinterval $\bar{T} \subset T$. Absolute continuity of $\mathbf{h}$ implies the existence of $\delta > 0$ such that whenever a finite sequence of pairwise disjoint subintervals $\{(a_k, b_k) : k = 1, \ldots, q\}$ of $\bar{T}$ satisfies $\sum_{k=1}^{q}(b_k - a_k) < \delta$ then $\sum_{k=1}^{q} \|\mathbf{h}(b_k) - \mathbf{h}(a_k)\| < \epsilon / L_{\mathbf{g}}$. It follows that

$$\sum_{k=1}^{q} \|\mathbf{f}(b_k) - \mathbf{f}(a_k)\| = \sum_{k=1}^{q} \|\mathbf{g}(\mathbf{h}(b_k)) - \mathbf{g}(\mathbf{h}(a_k))\| \leq L_{\mathbf{g}} \sum_{k=1}^{q} \|\mathbf{h}(b_k) - \mathbf{h}(a_k)\| < \epsilon.$$

□

**Definition 2.6.** Given an open set $X \subset \mathbb{R}^n$ and a function $\mathbf{f} : X \to \mathbb{R}^m$, $\mathbf{f}$ is said to be (Fréchet) *differentiable at* $\mathbf{x} \in X$ if there exists a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ that satisfies

$$\mathbf{0}_m = \lim_{\boldsymbol{\alpha} \to \mathbf{0}_n} \frac{\mathbf{f}(\mathbf{x} + \boldsymbol{\alpha}) - (\mathbf{f}(\mathbf{x}) + \mathbf{A}\boldsymbol{\alpha})}{\|\boldsymbol{\alpha}\|}.$$

In this case, the matrix $\mathbf{A}$ is uniquely described by the above equation and is called the *Jacobian matrix*, denoted by $\mathbf{Jf}(\mathbf{x}) \in \mathbb{R}^{m \times n}$. The function $\mathbf{f}$ is said to be *differentiable on* $X$ if it is differentiable at each point $\mathbf{x} \in X$.

**Definition 2.7.** Given an open set $X \subset \mathbb{R}^n$ and a function $\mathbf{f} : X \to \mathbb{R}^m$, $\mathbf{f}$ is said to be *continuously differentiable ($C^1$) at* $\mathbf{x} \in X$ if $\mathbf{f}$ is differentiable on a neighborhood $N(\mathbf{x}) \subset X$ of $\mathbf{x}$ and $\mathbf{Jf} : N(\mathbf{x}) \to \mathbb{R}^{m \times n}$ is continuous on $N(\mathbf{x})$. The function $\mathbf{f}$ is said to be *continuously differentiable ($C^1$) on* $X$ if $\mathbf{f}$ is $C^1$ at each point $\mathbf{x} \in X$.

**Definition 2.8.** [18] Given an open set $X \subset \mathbb{R}^n$ and a function $\mathbf{f} : X \to \mathbb{R}^m$, $\mathbf{f}$ is said to be *piecewise differentiable ($PC^1$)* at $\mathbf{x} \in X$ if there exist a neighborhood $N(\mathbf{x}) \subset X$ of $\mathbf{x}$ and a finite collection of $C^1$ functions on $N(\mathbf{x})$, $\{\mathbf{f}_{(1)}, \ldots, \mathbf{f}_{(k)}\}$, such that $\mathbf{f}$ is continuous on $N(\mathbf{x})$ and

$$\mathbf{f}(\boldsymbol{\eta}) \in \{\mathbf{f}_{(i)}(\boldsymbol{\eta}) : i \in \{1, \ldots, k\}\}, \quad \forall \boldsymbol{\eta} \in N(\mathbf{x}).$$

$\mathbf{f}$ is said to be $PC^1$ *on* $X$ if $\mathbf{f}$ is $PC^1$ at each point $\mathbf{x} \in X$.

**Remark 2.9.** Let $X \subset \mathbb{R}^n$ be open and $\mathbf{f} : X \to \mathbb{R}^m$. If $\mathbf{f}$ is $PC^1$ at $\mathbf{x} \in X$ then $\mathbf{f}$ is Lipschitz continuous on a neighborhood of $\mathbf{x}$. If $\mathbf{f}$ is $PC^1$ on $X$ then $\mathbf{f}$ is locally Lipschitz continuous on $X$.

Equivalent to the well-ordering theorem, Zorn's Lemma can be stated as follows (see [50, 51]).

**Definition 2.10.** Given a set $A$, a *partial order in* $A$, denoted by $\preceq$, is a relation between elements of $A$ satisfying reflexivity ($a \preceq a$ for all $a \in A$), antisymmetry ($a \preceq b$ and $b \preceq a$ imply $a = b$), and transitivity ($a \preceq c$ for any $a, b, c \in A$ satisfying $a \preceq b$ and $b \preceq c$). A *total order in* $A$ (or *chain in* $A$) is a partial order in $A$ which also satisfies comparability ($a \preceq b$ or $b \preceq a$ for all $a, b \in A$) . A *partially ordered set* is a set with a partial order. A *totally ordered set* is a set with a total order.

**Definition 2.11.** Let $A$ be a partially ordered set and $B \subset A$. The element $u \in A$ is called an *upper bound on* $B$ if $b \preceq u$ for every $b \in B$. The element $m \in A$ is called a *maximal element of* $A$ if $m = a$ for every $a \in A$ satisfying $m \preceq a$.

**Lemma 2.12. (Zorn's Lemma)** A nonempty partially ordered set in which every nonempty totally ordered subset has an upper bound contains maximal elements.

### 2.2. Generalized Derivatives

Let $X \subset \mathbb{R}^n$ be open and $\mathbf{f} : X \to \mathbb{R}^m$ be locally Lipschitz continuous on $X$. It follows that $\mathbf{f}$ is differentiable at each point $\mathbf{x} \in X \setminus Z_\mathbf{f}$, where $Z_\mathbf{f} \subset X$ has zero (Lebesgue) measure, by Rademacher's Theorem. Clarke [16] established the following definitions and results concerning generalized derivatives.

**Definition 2.13.** The *B-subdifferential* of $\mathbf{f}$ at $\mathbf{x} \in X$ is defined as

$$\partial_\mathrm{B}\mathbf{f}(\mathbf{x}) := \left\{ \lim_{i\to\infty} \mathbf{Jf}(\mathbf{x}_{(i)}) : \lim_{i\to\infty} \mathbf{x}_{(i)} = \mathbf{x}, \quad \mathbf{x}_{(i)} \in X \setminus Z_\mathbf{f}, \forall i \in \mathbb{N} \right\}.$$

**Definition 2.14.** The Clarke (generalized) *Jacobian* of $\mathbf{f}$ at $\mathbf{x} \in X$ is defined as $\partial\mathbf{f}(\mathbf{x}) := \mathrm{conv}\, \partial_\mathrm{B}\mathbf{f}(\mathbf{x})$.

**Remark 2.15.** For a point $\mathbf{x} \in X$, $\partial_\mathrm{B}\mathbf{f}(\mathbf{x})$ is necessarily nonempty and compact, while $\partial\mathbf{f}(\mathbf{x})$ is necessarily nonempty, compact, and convex. If $\mathbf{f}$ is differentiable at $\mathbf{x} \in X$ then $\mathbf{Jf}(\mathbf{x}) \in \partial\mathbf{f}_\mathrm{B}(\mathbf{x})$. If $\mathbf{f}$ is $C^1$ at $\mathbf{x}$ then $\partial\mathbf{f}(\mathbf{x}) = \partial_\mathrm{B}\mathbf{f}(\mathbf{x}) = \{\mathbf{Jf}(\mathbf{x})\}$.

**Definition 2.16.** Let $n_x, n_y, n_z, n_q \in \mathbb{N}$, $W \subset \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_z}$ be open, and $\mathbf{g} : W \to \mathbb{R}^{n_q}$ be Lipschitz continuous on a neighborhood of $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in W$. The Clarke (generalized) *Jacobian projections* of $\mathbf{g}$ at $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ are defined as

$$\begin{aligned}
\pi_1\partial\mathbf{g}(\mathbf{x}, \mathbf{y}, \mathbf{z}) &:= \left\{\mathbf{M} \in \mathbb{R}^{n_q \times n_x} : \exists [\mathbf{M} \quad \mathbf{N}_1 \quad \mathbf{N}_2] \in \partial\mathbf{g}(\mathbf{x}, \mathbf{y}, \mathbf{z})\right\}, \\
\pi_2\partial\mathbf{g}(\mathbf{x}, \mathbf{y}, \mathbf{z}) &:= \left\{\mathbf{M} \in \mathbb{R}^{n_q \times n_y} : \exists [\mathbf{N}_1 \quad \mathbf{M} \quad \mathbf{N}_2] \in \partial\mathbf{g}(\mathbf{x}, \mathbf{y}, \mathbf{z})\right\}, \\
\pi_{2,3}\partial\mathbf{g}(\mathbf{x}, \mathbf{y}, \mathbf{z}) &:= \left\{[\mathbf{M}_1 \quad \mathbf{M}_2] \in \mathbb{R}^{n_q \times (n_y + n_z)} : \exists [\mathbf{N} \quad \mathbf{M}_1 \quad \mathbf{M}_2] \in \partial\mathbf{g}(\mathbf{x}, \mathbf{y}, \mathbf{z})\right\},
\end{aligned}$$

with $\pi_3\partial\mathbf{g}(\mathbf{x}, \mathbf{y}, \mathbf{z})$, $\pi_{1,2}\partial\mathbf{g}(\mathbf{x}, \mathbf{y}, \mathbf{z})$, and $\pi_{1,3}\partial\mathbf{g}(\mathbf{x}, \mathbf{y}, \mathbf{z})$ defined similarly.

**Remark 2.17.** The Clarke Jacobian projections of $\mathbf{g}$ at $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ are equal to the projections of the Clarke Jacobian onto the appropriate subspace. The notation conventions in Definition 2.16 are chosen since the arguments of $\mathbf{g}$ eliminate any ambiguity. If $\mathbf{g}$ is $C^1$ at $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ then the Clarke Jacobian projections simplify to the partial derivatives [16]; for example, $\pi_1\partial\mathbf{g}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \left\{\frac{\partial\mathbf{g}}{\partial\mathbf{x}}(\mathbf{x}, \mathbf{y}, \mathbf{z})\right\}$ and

$$\pi_{2,3}\partial\mathbf{g}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \left\{\left[\frac{\partial\mathbf{g}}{\partial\mathbf{y}}(\mathbf{x}, \mathbf{y}, \mathbf{z}) \quad \frac{\partial\mathbf{g}}{\partial\mathbf{z}}(\mathbf{x}, \mathbf{y}, \mathbf{z})\right]\right\}.$$

## 3. Nonsmooth Implicit Function Theorems

Lemmata pertaining to generalized Jacobian projections are set out, culminating in a nonsmooth extended implicit function theorem.

### 3.1. Generalized Jacobian Projections of Maximal Rank

Intermediate results are provided for use in constructing nonsmooth implicit functions from nonsmooth inverse functions.

**Lemma 3.1.** Let $W \subset \mathbb{R}^n \times \mathbb{R}^m$ be open, $\mathbf{g} : W \to \mathbb{R}^m$ be Lipschitz continuous on a neighborhood $N \subset W$ of $(\mathbf{x}^*, \mathbf{y}^*) \in W$, and $\mathbf{f} : W \to \mathbb{R}^n \times \mathbb{R}^m : (\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{x}, \mathbf{g}(\mathbf{x}, \mathbf{y}))$. Then $\pi_2\partial\mathbf{g}(\mathbf{x}^*, \mathbf{y}^*)$ is of maximal rank if and only if $\partial\mathbf{f}(\mathbf{x}^*, \mathbf{y}^*)$ is of maximal rank.

*Proof.* Let $Z_\mathbf{g} \subset N$ denote the zero-measure subset of $N$ on which $\mathbf{g}$ is not differentiable, which is equal to the set on which $\mathbf{f}$ is not differentiable and has zero measure by Rademacher's Theorem. First suppose that $\pi_2\partial\mathbf{g}(\mathbf{x}^*, \mathbf{y}^*)$ is of maximal rank. If the result were not true, then there exists

$$\mathbf{H}^* \in \partial\mathbf{f}(\mathbf{x}^*, \mathbf{y}^*) = \left\{\sum_{i=1}^{k} \lambda_i\mathbf{H}_i : k \in \mathbb{N}, \sum_{i=1}^{k} \lambda_i = 1, \mathbf{H}_i \in \partial_\mathrm{B}\mathbf{f}(\mathbf{x}^*, \mathbf{y}^*), \lambda_i \geq 0\right\}$$

such that $\mathbf{H}^*$ is singular. That is, there exist $k^* \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_{k^*} \geq 0$, and $\mathbf{H}_1, \ldots, \mathbf{H}_{k^*} \in \partial_\mathrm{B}\mathbf{f}(\mathbf{x}^*, \mathbf{y}^*)$ such that $\sum_{i=1}^{k^*} \lambda_i = 1$ and $\sum_{i=1}^{k^*} \lambda_i\mathbf{H}_i = \mathbf{H}^*$ is singular. By definition of the B-subdifferential, for each $i \in \{1, \ldots, k^*\}$, there exists a sequence of vectors $\{(\mathbf{x}_{(i_j)}, \mathbf{y}_{(i_j)})\}$ such that $(\mathbf{x}_{(i_j)}, \mathbf{y}_{(i_j)}) \in N \setminus Z_{\mathbf{g}}$ for each $j \in \mathbb{N}$, $\lim_{j\to\infty}(\mathbf{x}_{(i_j)}, \mathbf{y}_{(i_j)}) = (\mathbf{x}^*, \mathbf{y}^*)$ and

$$\lim_{j\to\infty} \mathbf{Jf}(\mathbf{x}_{(i_j)}, \mathbf{y}_{(i_j)}) = \lim_{j\to\infty} \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n\times m} \\ \frac{\partial \mathbf{g}}{\partial \mathbf{x}}(\mathbf{x}_{(i_j)}, \mathbf{y}_{(i_j)}) & \frac{\partial \mathbf{g}}{\partial \mathbf{y}}(\mathbf{x}_{(i_j)}, \mathbf{y}_{(i_j)}) \end{bmatrix} = \mathbf{H}_i.$$

Thus,

$$\mathbf{H}^* = \sum_{i=1}^{k^*} \lambda_i \mathbf{H}_i = \sum_{i=1}^{k^*} \lambda_i \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n\times m} \\ \mathbf{A}_i & \mathbf{B}_i \end{bmatrix} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n\times m} \\ \sum_{i=1}^{k^*} \lambda_i \mathbf{A}_i & \sum_{i=1}^{k^*} \lambda_i \mathbf{B}_i \end{bmatrix}$$

where

$$\mathbf{A}_i := \lim_{j\to\infty} \frac{\partial \mathbf{g}}{\partial \mathbf{x}}(\mathbf{x}_{(i_j)}, \mathbf{y}_{(i_j)}), \quad \mathbf{B}_i := \lim_{j\to\infty} \frac{\partial \mathbf{g}}{\partial \mathbf{y}}(\mathbf{x}_{(i_j)}, \mathbf{y}_{(i_j)}), \quad \forall i \in \{1, \ldots, k^*\}.$$

Since $\mathbf{H}^*$ is singular, it must hold that $\sum_{i=1}^{k^*} \lambda_i \mathbf{B}_i$ is singular. However, $\sum_{i=1}^{k^*} \lambda_i \mathbf{B}_i \in \pi_2 \partial \mathbf{g}(\mathbf{x}^*, \mathbf{y}^*)$ since $[\mathbf{A}_i \quad \mathbf{B}_i] \in \partial_{\mathrm{B}} \mathbf{g}(\mathbf{x}^*, \mathbf{y}^*)$ for each $i \in \{1, \ldots, k^*\}$ implies that

$$\sum_{i=1}^{k^*} \lambda_i [\mathbf{A}_i \quad \mathbf{B}_i] = \begin{bmatrix} \sum_{i=1}^{k^*} \lambda_i \mathbf{A}_i & \sum_{i=1}^{k^*} \lambda_i \mathbf{B}_i \end{bmatrix} \in \partial \mathbf{g}(\mathbf{x}^*, \mathbf{y}^*).$$

This contradicts the assumption that $\pi_2 \partial \mathbf{g}(\mathbf{x}^*, \mathbf{y}^*)$ is of maximal rank.

Next, suppose that $\partial \mathbf{f}(\mathbf{x}^*, \mathbf{y}^*)$ is of maximal rank. If the result is not true, then there exists $\mathbf{B}^* \in \pi_2 \partial \mathbf{g}(\mathbf{x}^*, \mathbf{y}^*)$ and $\mathbf{A}^* \in \mathbb{R}^{m\times n}$ such that $\mathbf{B}^*$ is singular and $[\mathbf{A}^* \quad \mathbf{B}^*] \in \partial \mathbf{g}(\mathbf{x}^*, \mathbf{y}^*)$. It follows that there exist $k^* \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_{k^*} \geq 0$, and $[\mathbf{A}_1 \quad \mathbf{B}_1], [\mathbf{A}_2 \quad \mathbf{B}_2], \ldots, [\mathbf{A}_{k^*} \quad \mathbf{B}_{k^*}] \in \partial_{\mathrm{B}} \mathbf{g}(\mathbf{x}^*, \mathbf{y}^*)$ such that $\sum_{i=1}^{k^*} \lambda_i = 1$ and

$$[\mathbf{A}^* \quad \mathbf{B}^*] = \sum_{i=1}^{k^*} \lambda_i [\mathbf{A}_i \quad \mathbf{B}_i].$$

The definition of the B-subdifferential implies that, for each $i \in \{1, \ldots, k^*\}$, there exists a sequence of vectors $\{(\mathbf{x}_{(i_j)}, \mathbf{y}_{(i_j)})\}$ such that $(\mathbf{x}_{(i_j)}, \mathbf{y}_{(i_j)}) \in N \setminus Z_{\mathbf{g}}$ for each $j \in \mathbb{N}$, $\lim_{j\to\infty}(\mathbf{x}_{(i_j)}, \mathbf{y}_{(i_j)}) = (\mathbf{x}^*, \mathbf{y}^*)$ and

$$[\mathbf{A}_i \quad \mathbf{B}_i] = \lim_{j\to\infty} \begin{bmatrix} \frac{\partial \mathbf{g}}{\partial \mathbf{x}}(\mathbf{x}_{(i_j)}, \mathbf{y}_{(i_j)}) & \frac{\partial \mathbf{g}}{\partial \mathbf{y}}(\mathbf{x}_{(i_j)}, \mathbf{y}_{(i_j)}) \end{bmatrix}.$$

Therefore,

$$\lim_{j\to\infty} \mathbf{Jf}(\mathbf{x}_{(i_j)}, \mathbf{y}_{(i_j)}) = \lim_{j\to\infty} \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n\times m} \\ \frac{\partial \mathbf{g}}{\partial \mathbf{x}}(\mathbf{x}_{(i_j)}, \mathbf{y}_{(i_j)}) & \frac{\partial \mathbf{g}}{\partial \mathbf{y}}(\mathbf{x}_{(i_j)}, \mathbf{y}_{(i_j)}) \end{bmatrix} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n\times m} \\ \mathbf{A}_i & \mathbf{B}_i \end{bmatrix},$$

from which it follows that

$$\begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n\times m} \\ \sum_{i=1}^{k^*} \lambda_i \mathbf{A}_i & \sum_{i=1}^{k^*} \lambda_i \mathbf{B}_i \end{bmatrix} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n\times m} \\ \mathbf{A}^* & \mathbf{B}^* \end{bmatrix} \in \partial \mathbf{f}(\mathbf{x}^*, \mathbf{y}^*).$$

This contradicts the assumption that $\partial \mathbf{f}(\mathbf{x}^*, \mathbf{y}^*)$ is of maximal rank. □

For our purposes, an abridged restatement of Lemma 7.5.2 in [17] is given.

**Lemma 3.2.** Let $X \subset \mathbb{R}^n$ be open and $\boldsymbol{\Gamma}$ be a compact-valued, upper semicontinuous set-valued mapping from $X$ to $\mathbb{R}^{n\times n}$. Let $\mathbf{x}^* \in X$ and suppose that $\boldsymbol{\Gamma}(\mathbf{x}^*)$ is of maximal rank. Then there exist $\rho > 0$ such that $B_\rho(\mathbf{x}^*) \subset X$ and $\boldsymbol{\Gamma}(\mathbf{x})$ is of maximal rank for all $\mathbf{x} \in B_\rho(\mathbf{x}^*)$.

The above finding is extended to the constructions in Lemma 3.1 as follows.

**Lemma 3.3.** Let $W \subset \mathbb{R}^n \times \mathbb{R}^m$ be open, $\Omega \subset W$ be compact, and $\mathbf{g} : W \to \mathbb{R}^m$ be locally Lipschitz continuous on $W$. Let $\mathbf{f} : W \to \mathbb{R}^n \times \mathbb{R}^m : (\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{x}, \mathbf{g}(\mathbf{x}, \mathbf{y}))$. Suppose that $\pi_2 \partial \mathbf{g}(\mathbf{x}, \mathbf{y})$ is of maximal rank for all $(\mathbf{x}, \mathbf{y}) \in \Omega$ or $\partial \mathbf{f}(\mathbf{x}, \mathbf{y})$ is of maximal rank for all $(\mathbf{x}, \mathbf{y}) \in \Omega$. Then there exists $\rho > 0$ such that $\pi_2 \partial \mathbf{g}(\mathbf{x}, \mathbf{y})$ and $\partial \mathbf{f}(\mathbf{x}, \mathbf{y})$ are of maximal rank for all $(\mathbf{x}, \mathbf{y}) \in B_\rho(\Omega) \subset W$.

*Proof.* For any $(\mathbf{x}, \mathbf{y}) \in \Omega$, $\pi_2 \partial \mathbf{g}(\mathbf{x}, \mathbf{y})$ is of maximal rank if and only if $\partial \mathbf{f}(\mathbf{x}, \mathbf{y})$ is of maximal rank by Lemma 3.1. Suppose then, without loss of generality, that $\pi_2 \partial \mathbf{g}(\mathbf{x}, \mathbf{y})$ is of maximal rank. Since $W$ is open and $\Omega$ is compact, there exists $n^* \in \mathbb{N}$ such that $B_{1/n^*}(\Omega) \subset W$. If the claim does not hold, then for each $n \in \mathbb{N}$ such that $n \geq n^*$, there exists $(\mathbf{x}_{(n)}, \mathbf{y}_{(n)}) \in B_{1/n}(\Omega) \setminus \Omega$ and $\mathbf{B}_{(n)} \in \pi_2 \partial \mathbf{g}(\mathbf{x}_{(n)}, \mathbf{y}_{(n)})$ such that $\mathbf{B}_{(n)}$ is singular. The sequence $\{(\mathbf{x}_{(n)}, \mathbf{y}_{(n)})\}$ must have an accumulation point of the form $(\mathbf{x}^*, \mathbf{y}^*) \in \Omega$. Lemma 3.2 implies the existence of $\delta > 0$ such that $\partial \mathbf{f}(\mathbf{x}, \mathbf{y})$ is of maximal rank for all $(\mathbf{x}, \mathbf{y}) \in B_\delta(\mathbf{x}^*, \mathbf{y}^*) \subset W$. By application of Lemma 3.1 to each point in $B_\delta(\mathbf{x}^*, \mathbf{y}^*)$, $\pi_2 \partial \mathbf{g}(\mathbf{x}, \mathbf{y})$ is of maximal rank for all $(\mathbf{x}, \mathbf{y}) \in B_\delta(\mathbf{x}^*, \mathbf{y}^*)$. However, there exists $\widetilde{n} \in \mathbb{N}$ such that $\widetilde{n} \geq n^*$ and $(\mathbf{x}_{(\widetilde{n})}, \mathbf{y}_{(\widetilde{n})}) \in B_\delta(\mathbf{x}^*, \mathbf{y}^*)$, implying that $\mathbf{B}_{(\widetilde{n})} \in \pi_2 \partial \mathbf{g}(\mathbf{x}_{(\widetilde{n})}, \mathbf{y}_{(\widetilde{n})})$, a contradiction. Therefore, there exists $\rho > 0$ such that $\pi_2 \partial \mathbf{g}(\mathbf{x}, \mathbf{y})$ is of maximal rank for all $(\mathbf{x}, \mathbf{y}) \in B_\rho(\Omega) \subset W$. $\partial \mathbf{f}(\mathbf{x}, \mathbf{y})$ being of maximal rank for all $(\mathbf{x}, \mathbf{y}) \in B_\rho(\Omega)$ follows by repeated application of Lemma 3.1 to each point in $B_\rho(\Omega)$. $\square$

*3.2. Lipschitzian Extended Implicit Functions*

Extending the classical local inverse and implicit function theorems (see, e.g., [52]), Clarke showed that a Lipschitzian function has a local inverse near a point if its Clarke Jacobian is of maximal rank at said point (Theorem 7.1.1 in [16]).

**Theorem 3.4.** Let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ be Lipschitz continuous on a neighborhood of $\mathbf{x}^* \in \mathbb{R}^n$ and suppose that $\partial \mathbf{f}(\mathbf{x}^*)$ is of maximal rank. Then there exist neighborhoods $N(\mathbf{x}^*) \subset \mathbb{R}^n$ and $N(\mathbf{y}^*) \subset \mathbb{R}^n$ of $\mathbf{x}^*$ and $\mathbf{y}^* := \mathbf{f}(\mathbf{x}^*)$, respectively, and a function $\mathbf{f}^{-1} : N(\mathbf{y}^*) \to \mathbb{R}^n$ that is Lipschitz continuous on $N(\mathbf{y}^*)$ and satisfies $\mathbf{f}^{-1}(\mathbf{f}(\mathbf{x})) = \mathbf{x}$ for every $\mathbf{x} \in N(\mathbf{x}^*)$ and $\mathbf{f}(\mathbf{f}^{-1}(\mathbf{y})) = \mathbf{y}$ for every $\mathbf{y} \in N(\mathbf{y}^*)$.

Theorem 3.4 illustrates that the maximal rankness of the Clarke Jacobian is a sufficient condition for a function to be a local Lipschitz homeomorphism at a point. However, it should be noted that such a condition is not necessary (a counterexample was provided by Kummer [41]). As a corollary to Theorem 3.4, Clarke also provided a local nonsmooth implicit function theorem (Corollary to Theorem 7.1.1 in [16]), which is restated slightly here by virtue of the proof of Theorem 7.1.1 in [16] and Lemma 3.3 (with $\Omega$ equal to a singleton).

**Theorem 3.5.** Let $W \subset \mathbb{R}^n \times \mathbb{R}^m$ be open and $\mathbf{g} : W \to \mathbb{R}^m$ be Lipschitz continuous on a neighborhood of $(\mathbf{x}^*, \mathbf{y}^*) \in W$. Suppose that $\mathbf{g}(\mathbf{x}^*, \mathbf{y}^*) = \mathbf{0}_m$ and $\pi_2 \partial \mathbf{g}(\mathbf{x}^*, \mathbf{y}^*)$ is of maximal rank. Then there exist neighborhoods $N(\mathbf{x}^*) \subset \pi_x W$ and $N(\mathbf{x}^*, \mathbf{y}^*) \subset W$ of $\mathbf{x}^*$ and $(\mathbf{x}^*, \mathbf{y}^*)$, respectively, and a Lipschitz continuous function $\mathbf{r} : N(\mathbf{x}^*) \to \mathbb{R}^m$ such that $\pi_2 \partial \mathbf{g}(\mathbf{x}, \mathbf{y})$ is of maximal rank for all $(\mathbf{x}, \mathbf{y}) \in N(\mathbf{x}^*, \mathbf{y}^*)$ and, for each $\mathbf{x} \in N(\mathbf{x}^*)$, $(\mathbf{x}, \mathbf{r}(\mathbf{x}))$ is the unique vector in $N(\mathbf{x}^*, \mathbf{y}^*)$ satisfying $\mathbf{g}(\mathbf{x}, \mathbf{r}(\mathbf{x})) = \mathbf{0}_m$.

Drawing upon the idea of patching together local implicit functions, along the lines of [46] for the $C^1$-case, an extended nonsmooth implicit function theorem is derived.

**Theorem 3.6.** Let $W \subset \mathbb{R}^n \times \mathbb{R}^m$ be open and $\mathbf{g} : W \to \mathbb{R}^m$ be locally Lipschitz continuous on $W$. Let $\Omega \subset W$ be a compact set such that each point $\mathbf{x} \in \pi_x \Omega$ is the projection of only one point $(\mathbf{x}, \mathbf{y}) \in \Omega$. Suppose that $\pi_2 \partial \mathbf{g}(\mathbf{x}, \mathbf{y})$ is of maximal rank for each $(\mathbf{x}, \mathbf{y}) \in \Omega$ and $\mathbf{g}(\Omega) = \{\mathbf{0}_m\}$. Then there exist $\delta, \rho > 0$ and a function $\mathbf{r} : B_\delta(\pi_x \Omega) \subset \pi_x W \to \mathbb{R}^m$ that is Lipschitz continuous on $B_\delta(\pi_x \Omega)$ such that $\pi_2 \partial \mathbf{g}(\mathbf{x}, \mathbf{y})$ is of maximal rank for all $(\mathbf{x}, \mathbf{y}) \in B_\rho(\Omega) \subset W$ and, for each $\mathbf{x} \in B_\delta(\pi_x \Omega)$, $(\mathbf{x}, \mathbf{r}(\mathbf{x}))$ is the unique vector in $B_\rho(\Omega)$ satisfying $\mathbf{g}(\mathbf{x}, \mathbf{r}(\mathbf{x})) = \mathbf{0}_m$.

*Proof.* We claim that there exists $\xi > 0$ for which $B_\xi(\Omega)$ is such that $B_\xi(\Omega) \subset W$ and if $\mathbf{g}(\boldsymbol{\eta}_x, \boldsymbol{\eta}_{y_1}) = \mathbf{g}(\boldsymbol{\eta}_x, \boldsymbol{\eta}_{y_2}) = \mathbf{0}_m$ for some $(\boldsymbol{\eta}_x, \boldsymbol{\eta}_{y_1}), (\boldsymbol{\eta}_x, \boldsymbol{\eta}_{y_2}) \in B_\xi(\Omega)$, then $\boldsymbol{\eta}_{y_1} = \boldsymbol{\eta}_{y_2}$. Since $W$ is open and $\Omega$ is compact, there exists $n^* \in \mathbb{N}$ such that $B_{1/n^*}(\Omega) \subset W$. If the claim does not hold, then for each $n \in \mathbb{N}$ such that $n \geq n^*$, there exist $(\mathbf{x}_{(n)}, \mathbf{y}_{(n)}), (\mathbf{x}_{(n)}, \mathbf{y}^*_{(n)}) \in B_{1/n}(\Omega) \setminus \Omega$ such that $\mathbf{y}_{(n)} \neq \mathbf{y}^*_{(n)}$ and $\mathbf{g}(\mathbf{x}_{(n)}, \mathbf{y}_{(n)}) = \mathbf{g}(\mathbf{x}_{(n)}, \mathbf{y}^*_{(n)}) = \mathbf{0}_m$. The sequences $\{(\mathbf{x}_{(n)}, \mathbf{y}_{(n)})\}$ and $\{(\mathbf{x}_{(n)}, \mathbf{y}^*_{(n)})\}$ must have accumulation points of the form $(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}), (\widetilde{\mathbf{x}}, \mathbf{y}^*) \in \Omega$, respectively. By assumption, $\widetilde{\mathbf{x}} \in \pi_x \Omega$ is the projection of a unique point in $\Omega$, that is, $\widetilde{\mathbf{y}} = \mathbf{y}^*$.

By Theorem 3.5, there exist neighborhoods $N(\widetilde{\mathbf{x}}) \subset \pi_x W$ and $N(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}) \subset W$ of $\widetilde{\mathbf{x}}$ and $(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})$, respectively, and a Lipschitz continuous function $\mathbf{r}_{\widetilde{\mathbf{x}}} : N(\widetilde{\mathbf{x}}) \to \mathbb{R}^m$ such that, for all $\mathbf{x} \in N(\widetilde{\mathbf{x}})$, $(\mathbf{x}, \mathbf{r}_{\widetilde{\mathbf{x}}}(\mathbf{x}))$ is the unique vector in $N(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})$ satisfying $\mathbf{g}(\mathbf{x}, \mathbf{r}_{\widetilde{\mathbf{x}}}(\mathbf{x})) = \mathbf{0}_m$. However, by the above arguments, there exists $\widetilde{n} \in \mathbb{N}$ such that $\widetilde{n} \geq n^*$ and $(\mathbf{x}_{(\widetilde{n})}, \mathbf{y}_{(\widetilde{n})}), (\mathbf{x}_{(\widetilde{n})}, \mathbf{y}^*_{(\widetilde{n})}) \in N(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})$ satisfy $\mathbf{g}(\mathbf{x}_{(\widetilde{n})}, \mathbf{y}_{(\widetilde{n})}) = \mathbf{g}(\mathbf{x}_{(\widetilde{n})}, \mathbf{y}^*_{(\widetilde{n})}) = \mathbf{0}_m$ with $\mathbf{y}_{(\widetilde{n})} \neq \mathbf{y}^*_{(\widetilde{n})}$, a contradiction.

Since $\pi_2\partial\mathbf{g}(\mathbf{x},\mathbf{y})$ is of maximal rank for each $(\mathbf{x},\mathbf{y}) \in \Omega$ and $\mathbf{g}(\Omega) = \{\mathbf{0}_m\}$, Theorem 3.5 can be applied to each point in $\Omega$ to furnish the collection of functions $\{\mathbf{r}_{\mathbf{x}} : \mathbf{x} \in \pi_x\Omega\}$ and neighborhoods $\{N(\mathbf{x}) \subset \pi_x W : \mathbf{x} \in \pi_x\Omega\}$, $\{N(\mathbf{x},\mathbf{y}) \subset W : (\mathbf{x},\mathbf{y}) \in \Omega\}$. For each $\mathbf{x} \in \pi_x\Omega$, there exists $\xi(\mathbf{x}) > 0$ such that $B_{\xi(\mathbf{x})}(\mathbf{x},\mathbf{r}_{\mathbf{x}}(\mathbf{x})) \subset N(\mathbf{x},\mathbf{r}_{\mathbf{x}}(\mathbf{x}))\cap B_\xi(\Omega)$. In addition, it is possible to choose $\alpha(\mathbf{x}) > 0$ sufficiently small such that $B_{\alpha(\mathbf{x})}(\mathbf{x}) \subset N(\mathbf{x})$ and $(\boldsymbol{\eta}_x, \mathbf{r}_{\mathbf{x}}(\boldsymbol{\eta}_x))$ is the unique vector in $B_{\xi(\mathbf{x})}(\mathbf{x},\mathbf{r}_{\mathbf{x}}(\mathbf{x}))$ satisfying $\mathbf{g}(\boldsymbol{\eta}_x, \mathbf{r}_{\mathbf{x}}(\boldsymbol{\eta}_x)) = \mathbf{0}_m$ for all $\boldsymbol{\eta}_x \in B_{\alpha(\mathbf{x})}(\mathbf{x})$.

Define the set

$$U := \bigcup_{\mathbf{x}\in\pi_x\Omega} B_{0.5\alpha(\mathbf{x})}(\mathbf{x}) \subset \pi_x W,$$

which is an open cover of $\pi_x\Omega$. Since $\Omega$ is compact, its projection onto $\mathbb{R}^n$ is also compact. Therefore, $U$ has a finite subcover: there exist $q \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_q > 0$, $\mathbf{x}_{(1)}, \ldots, \mathbf{x}_{(q)} \in \pi_x\Omega$ such that

$$\pi_x\Omega \subset \bigcup_{i\in\{1,\ldots,q\}} B_{0.5\alpha_i}(\mathbf{x}_{(i)}) =: U^* \subset \pi_x W.$$

Since $\pi_x\Omega$ is compact and $\pi_x W$ is open, there exists $\alpha \in (0, 0.5\min\{\alpha_1,\ldots,\alpha_q\}]$ such that $B_\alpha(\pi_x\Omega) \subset \pi_x W$. Choose any point $\mathbf{x}^* \in B_\alpha(\pi_x\Omega)$. Then $\mathbf{x}^* = \widetilde{\mathbf{x}} + \mathbf{x}_\alpha$ for some $\widetilde{\mathbf{x}} \in \pi_x\Omega$ and $\mathbf{x}_\alpha \in \{\boldsymbol{\eta}_x \in \mathbb{R}^n : \|\boldsymbol{\eta}_x\| < \alpha\}$. Since $\pi_x\Omega \subset U^*$, there exists a nonempty index set $\mathcal{P} \subset \{1,\ldots,q\}$ such that

$$\widetilde{\mathbf{x}} \in \bigcup_{i\in\mathcal{P}} B_{0.5\alpha_i}(\mathbf{x}_{(i)}).$$

For any $i \in \mathcal{P}$,

$$\|\mathbf{x}^* - \mathbf{x}_{(i)}\| = \|\widetilde{\mathbf{x}} + \mathbf{x}_\alpha - \mathbf{x}_{(i)}\| \le \|\widetilde{\mathbf{x}} - \mathbf{x}_{(i)}\| + \|\mathbf{x}_\alpha\| < 0.5\alpha_i + \alpha \le \alpha_i,$$

and so

$$\mathbf{x}^* \in \bigcup_{i\in\mathcal{P}} B_{\alpha_i}(\mathbf{x}_{(i)}).$$

Furthermore, for each $i \in \mathcal{P}$,

$$\mathbf{0}_m = \mathbf{g}(\mathbf{x}^*, \mathbf{r}_{\mathbf{x}_{(i)}}(\mathbf{x}^*)),$$

and $(\mathbf{x}^*, \mathbf{r}_{\mathbf{x}_{(i)}}(\mathbf{x}^*)) \in B_\xi(\Omega)$. Hence, $\mathbf{r}_{\mathbf{x}_{(i)}}(\mathbf{x}^*) = \mathbf{r}_{\mathbf{x}_{(j)}}(\mathbf{x}^*)$ for all $i, j \in \mathcal{P}$. It follows that the mapping

$$\mathbf{r} : B_\alpha(\pi_x\Omega) \to \mathbb{R}^m : \boldsymbol{\eta}_x \mapsto \mathbf{r}_{\mathbf{x}_{(i)}}(\boldsymbol{\eta}_x), \quad \text{if } \boldsymbol{\eta}_x \in B_{\alpha_i}(\mathbf{x}_{(i)}),$$

is well-defined.

Let $\hat{\mathbf{x}} \in B_\alpha(\pi_x\Omega)$. Then $\hat{\mathbf{x}} \in B_{\alpha_i}(\mathbf{x}_{(i)})$ for some $i \in \{1,\ldots,q\}$. Since $\mathbf{r} = \mathbf{r}_{\mathbf{x}_{(i)}}$ on $B_{\alpha_i}(\mathbf{x}_{(i)}) \cap B_\alpha(\pi_x\Omega)$, $\mathbf{r}$ is Lipschitz continuous on a neighborhood of $\hat{\mathbf{x}}$ and $(\hat{\mathbf{x}}, \mathbf{r}(\hat{\mathbf{x}})) = (\hat{\mathbf{x}}, \mathbf{r}_{\mathbf{x}_{(i)}}(\hat{\mathbf{x}}))$ is the unique vector in $B_\xi(\Omega) \subset W$ satisfying $\mathbf{g}(\hat{\mathbf{x}}, \mathbf{r}(\hat{\mathbf{x}})) = \mathbf{0}_m$. Hence, $\mathbf{r}$ is locally Lipschitz continuous on $B_\alpha(\pi_x\Omega)$, and is therefore Lipschitz continuous on the compact set $\bar{B}_{0.5\alpha}(\pi_x\Omega)$. By Lemma 3.3, there exists $\gamma > 0$ such that $\pi_2\partial\mathbf{g}(\mathbf{x},\mathbf{y})$ is of maximal rank for all $(\mathbf{x},\mathbf{y}) \in B_\gamma(\Omega) \subset W$. The result holds by choosing $\delta := 0.5\alpha$ and $\rho := \min\{\xi, \gamma\}$. □

**Remark 3.7.** The openness of the set $B_\delta(\pi_x\Omega)$ outlined in the statement of Theorem 3.6 is of importance: first, Theorem 3.5 is recovered from Theorem 3.6 when $\Omega$ is a singleton. Second, the implicit function $\mathbf{r}$ is furnished with an open domain of definition, which is of value for results in the present paper detailing equivalence of DAEs and ODEs on open and connected sets and potential application in other works (e.g., the openness is a requirement for its lexicographic smoothness [48]).

The next lemma is motivated by the need to demonstrate uniqueness of projections of sets for application of Theorem 3.6.

**Lemma 3.8.** Let $n_x, n_y, n_z \in \mathbb{N}$, $X \subset \mathbb{R}^{n_x}$, $\mathbf{f} : X \to \mathbb{R}^{n_y}$, and $\mathbf{g} : X \to \mathbb{R}^{n_z}$. Then each point in $\{(\mathbf{x}, \mathbf{f}(\mathbf{x})) : \mathbf{x} \in X\}$ is the projection of a unique point in $\{(\mathbf{x}, \mathbf{f}(\mathbf{x}), \mathbf{g}(\mathbf{x})) : \mathbf{x} \in X\}$.

*Proof.* Let $(\mathbf{x}^*, \mathbf{f}^*) \in \Lambda := \{(\mathbf{x}, \mathbf{f}(\mathbf{x})) : \mathbf{x} \in X\}$, which implies that $\mathbf{x}^* \in X$. Then, by construction, $\mathbf{f}^* = \mathbf{f}(\mathbf{x}^*)$ and

$$\{(\mathbf{x}^*, \mathbf{f}^*)\} = \{(\mathbf{x}^*, \mathbf{f}(\mathbf{x}^*))\} = \pi_{x,y}\{(\mathbf{x}^*, \mathbf{f}(\mathbf{x}^*), \mathbf{g}(\mathbf{x}^*))\}$$

where $(\mathbf{x}^*, \mathbf{f}(\mathbf{x}^*), \mathbf{g}(\mathbf{x}^*)) \in \Omega := \{(\mathbf{x}, \mathbf{f}(\mathbf{x}), \mathbf{g}(\mathbf{x})) : \mathbf{x} \in X\}$. That is, each point in $\Lambda$ is the projection of at least one point in $\Omega$. Suppose that there exists $(\widetilde{\mathbf{x}}, \widetilde{\mathbf{f}}, \widetilde{\mathbf{g}}) \in \Omega \setminus \{(\mathbf{x}^*, \mathbf{f}(\mathbf{x}^*), \mathbf{g}(\mathbf{x}^*))\}$ such that

$$\{(\mathbf{x}^*, \mathbf{f}^*)\} = \pi_{x,y}\{(\widetilde{\mathbf{x}}, \widetilde{\mathbf{f}}, \widetilde{\mathbf{g}})\}.$$

Then $\mathbf{x}^* = \widetilde{\mathbf{x}}$ and

$$(\widetilde{\mathbf{x}}, \widetilde{\mathbf{f}}, \widetilde{\mathbf{g}}) = (\widetilde{\mathbf{x}}, \mathbf{f}(\widetilde{\mathbf{x}}), \mathbf{g}(\widetilde{\mathbf{x}})) = (\mathbf{x}^*, \mathbf{f}(\mathbf{x}^*), \mathbf{g}(\mathbf{x}^*)),$$

which is a contradiction. □

## 4. Carathéodory Index-1 Semi-Explicit Differential-Algebraic Equations

Let $n_p, n_x, n_y \in \mathbb{N}$ and $n_t \equiv 1$ to make the exposition more intuitive. Let $D \subset \mathbb{R}^{n_t} \times \mathbb{R}^{n_p} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_y}$ be open and connected. Let $\mathbf{f} : D \to \mathbb{R}^{n_x}$, $\mathbf{g} : D \to \mathbb{R}^{n_y}$, and $\mathbf{f}_0 : \pi_p D \to \pi_x D$. Given $t_0 \in \pi_t D$, consider the following IVP in semi-explicit DAEs:

$$\dot{\mathbf{x}}(t, \mathbf{p}) = \mathbf{f}(t, \mathbf{p}, \mathbf{x}(t, \mathbf{p}), \mathbf{y}(t, \mathbf{p})), \tag{2a}$$
$$\mathbf{0}_{n_y} = \mathbf{g}(t, \mathbf{p}, \mathbf{x}(t, \mathbf{p}), \mathbf{y}(t, \mathbf{p})), \tag{2b}$$
$$\mathbf{x}(t_0, \mathbf{p}) = \mathbf{f}_0(\mathbf{p}), \tag{2c}$$

where $t$ is the independent variable and $\mathbf{p} \in \pi_p D$ is a vector of the problem parameters. The following assumptions are made regarding the right-hand side functions in (2b) and (2c).

**Assumption 4.1.** Suppose that $\mathbf{f}_0$ is locally Lipschitz continuous on $\pi_p D$.

**Assumption 4.2.** Suppose that $\mathbf{g}$ is locally Lipschitz continuous on $D$.

Under Assumption 4.2, notions of consistent initialization and generalized differential index of (2) are captured as follows (motivated by the $C^1$-case analogues found in [29]).

**Definition 4.3.** The *consistency set, initial consistency set, and regularity set of* (2) are given by, respectively,

$$\begin{aligned} G_{\mathrm{C}} &:= \{(t, \mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y) \in D : \mathbf{g}(t, \mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y) = \mathbf{0}_{n_y}\}, \\ G_{\mathrm{C},0} &:= \{(t, \mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y) \in G_{\mathrm{C}} : t = t_0, \boldsymbol{\eta}_x = \mathbf{f}_0(\mathbf{p})\}, \\ G_{\mathrm{R}} &:= \{(t, \mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y) \in D : \pi_4 \partial \mathbf{g}(t, \mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y) \text{ is of maximal rank}\}. \end{aligned}$$

Using the concept of regularity of a solution to DAEs introduced in [29], the following definitions are made regarding (2).

**Definition 4.4.** Let $T \subset \pi_t D$ be a connected set containing $t_0$ and $P \subset \pi_p D$. A mapping $\mathbf{z} \equiv (\mathbf{x}, \mathbf{y}) : T \times P \to \pi_{x,y} D$ is called a *solution of* (2) *on* $T \times P$ if, for each $\mathbf{p} \in P$, $\mathbf{z}(\cdot, \mathbf{p})$ is an absolutely continuous function which satisfies (2a) for almost every $t \in T$, (2b) for every $t \in T$, and (2c) at $t = t_0$. If, in addition,

$$\{(t, \mathbf{p}, \mathbf{x}(t, \mathbf{p}), \mathbf{y}(t, \mathbf{p})) : (t, \mathbf{p}) \in T \times P\} \subset G_{\mathrm{R}},$$

then $\mathbf{z}$ is called a *regular* solution of (2) on $T \times P$.

**Remark 4.5.** If $\mathbf{g}$ is $C^1$ on $D$ then $G_{\mathrm{R}} = \{(t, \mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y) \in D : \det \frac{\partial \mathbf{g}}{\partial \mathbf{y}}(t, \mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y) \neq 0\}$ and the property that (2) has differential index equal to one (see [1, 2]) for all $(t, \mathbf{p}) \in T \times P$ is implied by regularity.

**Example 4.6.** Consider the following IVP in semi-explicit DAEs:

$$\begin{aligned} \dot{x}(t, p) &= \mathrm{sign}(t - 0.5) + (1.5|1 - \eta_y|^{\frac{1}{3}} - 1)\mathrm{H}(t - 1), \\ 0 &= |x(t, p)| + |y(t, p)| - 1, \\ x(t_0, p) &= \min\{0, p\}, \end{aligned} \tag{3}$$

where the signum and Heaviside functions are defined as follows:

$$\text{sign} : \mathbb{R} \to \{-1, 0, 1\} : t \mapsto \begin{cases} 1, & \text{if } t > 0, \\ 0, & \text{if } t = 0, \\ -1, & \text{if } t < 0, \end{cases} \quad \text{and} \quad \text{H} : \mathbb{R} \to \{0, 1\} : t \mapsto \begin{cases} 1, & \text{if } t \geq 0, \\ 0, & \text{if } t < 0. \end{cases}$$

The right-hand side functions are given by

$$\begin{aligned} f &: \mathbb{R}^4 \to \mathbb{R} : (t, p, \eta_x, \eta_y) \mapsto \text{sign}(t - 0.5) + (1.5|1 - \eta_y|^{\frac{1}{3}} - 1)H(t - 1), \\ g &: \mathbb{R}^4 \to \mathbb{R} : (t, p, \eta_x, \eta_y) \mapsto |\eta_x| + |\eta_y| - 1, \\ f_0 &: \mathbb{R} \to \mathbb{R} : p \mapsto \min\{0, p\}. \end{aligned}$$

Given $(t, p, \eta_x, \eta_y) \in \mathbb{R}^4$,

$$\pi_4 \partial g(t, p, \eta_x, \eta_y) = \begin{cases} \{-1\}, & \text{if } \eta_y < 0, \\ [-1, 1], & \text{if } \eta_y = 0, \\ \{1\}, & \text{if } \eta_y > 0. \end{cases}$$

The consistency, initial consistency, and regularity sets are given by

$$\begin{aligned} G_{\text{C}} &= \{(t, p, \eta_x, \eta_y) \in \mathbb{R}^4 : |\eta_x| + |\eta_y| - 1 = 0\}, \\ G_{\text{C},0} &= \{(t, p, \eta_x, \eta_y) \in G_{\text{C}} : t = t_0, \eta_x = \min\{0, p\}\}, \\ G_{\text{R}} &= \{(t, p, \eta_x, \eta_y) \in \mathbb{R}^4 : \eta_y \neq 0\}. \end{aligned}$$

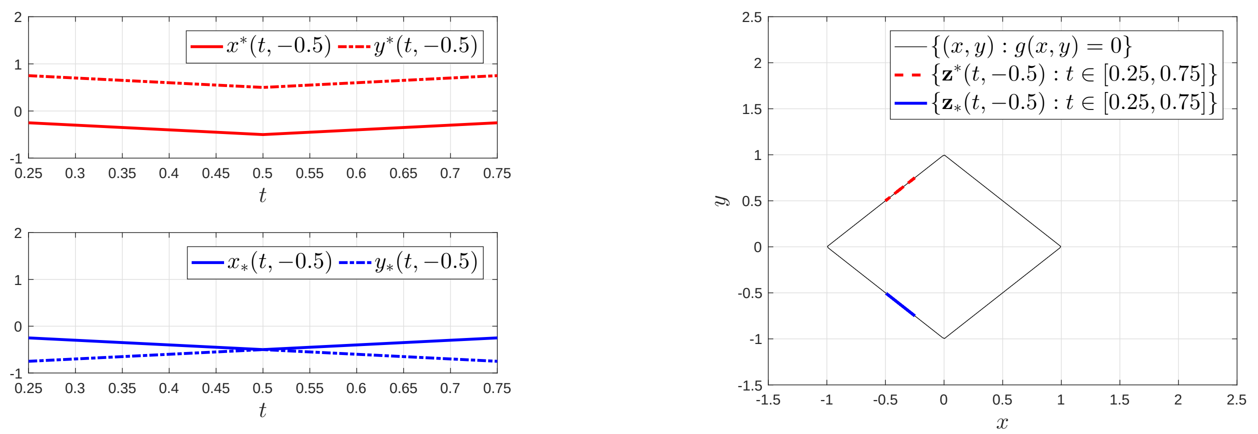
Suppose that $t_0 := 0.5$. There are two solutions of (3) on $[0.25, 0.75] \times \{-0.5\}$:

$$\mathbf{z}^* \equiv (x^*, y^*) : [0.25, 0.75] \times \{-0.5\} \to \mathbb{R}^2 : (t, p) \mapsto \begin{cases} (-t, 1 - t), & \text{if } t \in [0.25, 0.5], \\ (t - 1, t), & \text{if } t \in (0.5, 0.75], \end{cases}$$

and

$$\mathbf{z}_* \equiv (x_*, y_*) : [0.25, 0.75] \times \{-0.5\} \to \mathbb{R}^2 : (t, p) \mapsto \begin{cases} (-t, t - 1), & \text{if } t \in [0.25, 0.5], \\ (t - 1, -t), & \text{if } t \in (0.5, 0.75]. \end{cases}$$

$\mathbf{z}^*$ and $\mathbf{z}_*$ are both regular; $y^*(t, p) > 0$ for all $(t, p) \in [0.25, 0.75] \times \{-0.5\}$ and $y_*(t, p) < 0$ for all $(t, p) \in [0.25, 0.75] \times \{-0.5\}$. See Figure 2 for an illustration.




(a) $\mathbf{z}^*(t, -0.5)$ and $\mathbf{z}_*(t, -0.5)$ vs. $t$.

(b) Algebraic constraint and solution trajectories.

Figure 2: Graphs of Example 4.6.

The ambiguity apparent in Example 4.6 with respect to initialization is resolved through the next definition.

**Definition 4.7.** Let $\Omega_0 \subset G_{C,0}$. $\mathbf{z}$ is said to be a *(regular) solution of* (2) *on* $T \times P$ *through* $\Omega_0$ if $\mathbf{z}$ is a (regular) solution of (2) on $T \times P$ and, in addition,

$$\{(t_0, \mathbf{p}, \mathbf{x}(t_0, \mathbf{p}), \mathbf{y}(t_0, \mathbf{p})) : \mathbf{p} \in P\} = \Omega_0.$$

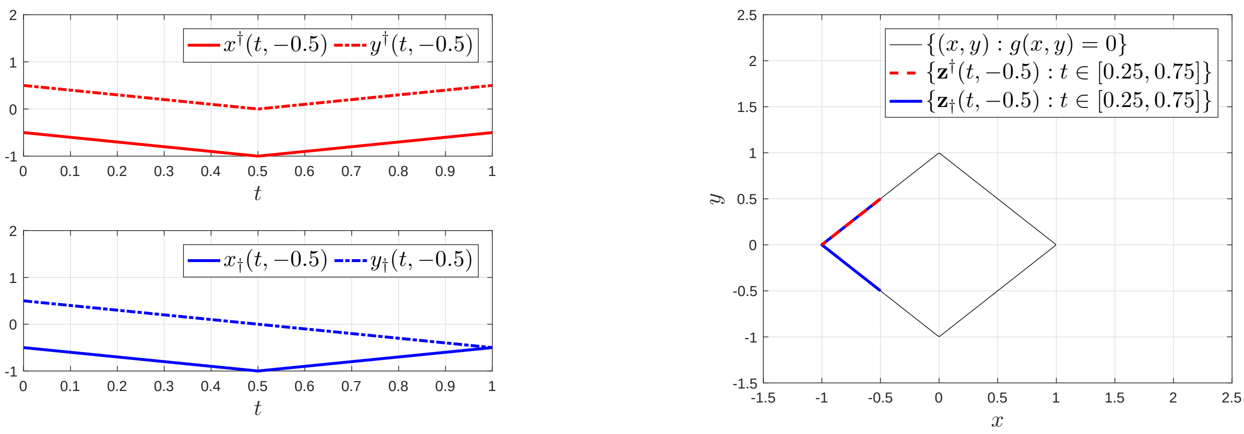
**Example 4.8.** Consider (3) and the mappings $\mathbf{z}^*$ and $\mathbf{z}_*$ outlined in Example 4.6. By inspection, $\mathbf{z}^*(0.5, -0.5) = (-0.5, 0.5)$ and $\mathbf{z}_*(0.5, -0.5) = (-0.5, -0.5)$. That is, $\mathbf{z}^*$ is a regular solution of (3) on $[0.25, 0.75] \times \{-0.5\}$ through $\{(0.5, -0.5, -0.5, 0.5)\}$ while $\mathbf{z}_*$ is a regular solution of (3) on $[0.25, 0.75] \times \{-0.5\}$ through $\{(0.5, -0.5, -0.5, -0.5)\}$.
There exist two solutions of (3) on $[0, 1] \times \{-0.5\}$ through $\{(0, -0.5, -0.5, 0.5)\}$:

$$\mathbf{z}^\dagger \equiv (x^\dagger, y^\dagger) : [0, 1] \times \{-0.5\} \to \mathbb{R}^2 : (t, p) \mapsto \begin{cases} (-t - 0.5, 0.5 - t), & \text{if } t \in [0, 0.5], \\ (t - 1.5, t - 0.5), & \text{if } t \in (0.5, 1], \end{cases}$$

and

$$\mathbf{z}_\dagger \equiv (x_\dagger, y_\dagger) : [0, 1] \times \{-0.5\} \to \mathbb{R}^2 : (t, p) \mapsto \begin{cases} (-t - 0.5, 0.5 - t), & \text{if } t \in [0, 0.5], \\ (t - 1.5, 0.5 - t), & \text{if } t \in (0.5, 1]. \end{cases}$$

Neither $\mathbf{z}^\dagger$ nor $\mathbf{z}_\dagger$ are regular since $y^\dagger(0.5, -0.5) = y_\dagger(0.5, -0.5) = 0$. See Figure 3 for an illustration.




(a) $\mathbf{z}^\dagger(t, -0.5)$ and $\mathbf{z}_\dagger(t, -0.5)$ vs. $t$.

(b) Algebraic constraint and solution trajectories.

Figure 3: Graphs of Example 4.8.

The following convention is used for highlighting augmented solution codomains.

**Definition 4.9.** Let $N \subset D$. $\mathbf{z}$ is said to be a *(regular) solution of* (2) *on* $T \times P$ *(through* $\Omega_0$*) in* $N$ if $\mathbf{z}$ is a (regular) solution of (2) on $T \times P$ (through $\Omega_0$) and, in addition,

$$\{(t, \mathbf{p}, \mathbf{x}(t, \mathbf{p}), \mathbf{y}(t, \mathbf{p})) : (t, \mathbf{p}) \in T \times P\} \subset N.$$

Uniqueness of solutions is characterized as follows.

**Definition 4.10.** Let $\mathbf{z}$ be a solution of (2) on $T \times P$ through $\Omega_0$. Then $\mathbf{z}$ is said to be *unique* if, given any other solution $\mathbf{z}^*$ of (2) on $T^* \times P^*$ through $\Omega_0^*$ satisfying $T \cap T^* \neq \{t_0\}$, $P \cap P^* \neq \emptyset$, and

$$\{(t_0, \mathbf{p}, \mathbf{z}(t_0, \mathbf{p})) : \mathbf{p} \in P \cap P^*\} = \{(t_0, \mathbf{p}, \mathbf{z}^*(t_0, \mathbf{p})) : \mathbf{p} \in P \cap P^*\},$$

$\mathbf{z}(t, \mathbf{p}) = \mathbf{z}^*(t, \mathbf{p})$ for all $(t, \mathbf{p}) \in (T \cap T^*) \times (P \cap P^*)$.

The sets $T$ and $T^*$ outlined in Definition 4.10 both contain $t_0$ by definition of a solution (Definition 4.4).

**Example 4.11.** Consider (3) and the mappings $\mathbf{z}^*$ and $\mathbf{z}^\dagger$ outlined in Examples 4.6 and 4.8, respectively. $\mathbf{z}^*$ is the unique regular solution of (3) on $[0.25, 0.75] \times \{-0.5\}$ through $\{(0.5, -0.5, -0.5, 0.5)\}$. $\mathbf{z}^\dagger$ is a solution of (3) on $[0, 1] \times \{-0.5\}$ through $\{(0, -0.5, -0.5, 0.5)\}$ but is not unique.

Define the following mappings:

$$\mathbf{w} : [-0.5, 1] \to \mathbb{R}^2 : t \mapsto \begin{cases} (-t, 1+t), & \text{if } t \in [-0.5, 0], \\ (-t, 1-t), & \text{if } t \in (0, 0.5], \\ (t-1, t), & \text{if } t \in (0.5, 1], \end{cases}$$

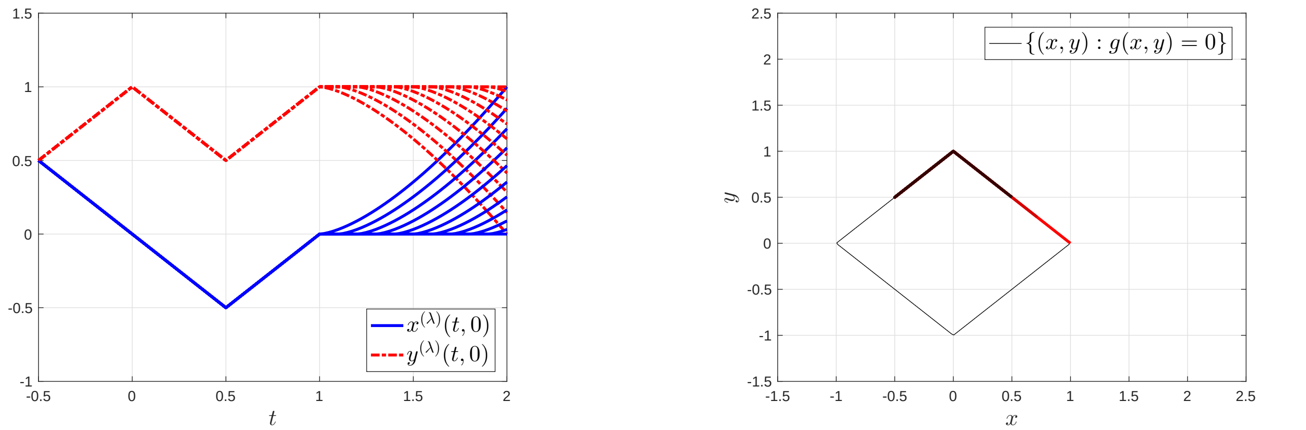
$$\mathbf{z}^{(0)} \equiv (x^{(0)}, y^{(0)}) : [-0.5, 2] \times \{0\} \to \mathbb{R}^2 : (t, p) \mapsto \begin{cases} \mathbf{w}(t), & \text{if } t \in [-0.5, 1], \\ ((t-1)^{\frac{3}{2}}, 1 - (t-1)^{\frac{3}{2}}), & \text{if } t \in (1, 2], \end{cases}$$

$$\mathbf{z}^{(1)} \equiv (x^{(1)}, y^{(1)}) : [-0.5, 2] \times \{0\} \to \mathbb{R}^2 : (t, p) \mapsto \begin{cases} \mathbf{w}(t), & \text{if } t \in [-0.5, 1], \\ (0, 1), & \text{if } t \in (1, 2], \end{cases}$$

and, for each $\lambda \in (0, 1)$,

$$\mathbf{z}^{(\lambda)} \equiv (x^{(\lambda)}, y^{(\lambda)}) : [-0.5, 2] \times \{0\} \to \mathbb{R}^2 : (t, p) \mapsto \begin{cases} \mathbf{w}(t), & \text{if } t \in [-0.5, 1], \\ (0, 1), & \text{if } t \in (1, 1+\lambda], \\ ((t-1-\lambda)^{\frac{3}{2}}, 1 - (t-1-\lambda)^{\frac{3}{2}}), & \text{if } t \in (1+\lambda, 2]. \end{cases}$$

There are an uncountable number of regular solutions of (3) on $[-0.5, 2] \times \{0\}$ through $\{(0, 0, 0, 1)\}$ (namely, each function in $\{\mathbf{z}^{(\lambda)} : \lambda \in (0, 1]\}$). $\mathbf{z}^{(0)}$ is also a solution of (3) on $[-0.5, 2] \times \{0\}$ through $\{(0, 0, 0, 1)\}$ but is not regular since $y^{(0)}(2, 0) = 0$. See Figure 4 for an illustration.




(a) $\mathbf{z}^{(\lambda)}(t, 0)$ vs. $t$ for various values of $0 \le \lambda \le 1$.

(b) Algebraic constraint and solution trajectories $\{\mathbf{z}^{(\lambda)}(t, 0) : -0.5 \le t \le 2\}$ (the lightest shade of red corresponds to $\lambda = 0$ while the darkest shade corresponds to $\lambda = 1$).

Figure 4: Graphs of Example 4.11.

### *4.1. Consistent Initialization*

Recalling that $t_0 \in \pi_t D$ is given, consider the nonlinear equation system associated with (2) at initialization:

$$\begin{aligned} \dot{\mathbf{x}}_{\text{init}} &= \mathbf{f}_{t_0}(\mathbf{p}, \mathbf{x}_{\text{init}}, \mathbf{y}_{\text{init}}), \\ \mathbf{0}_{n_y} &= \mathbf{g}_{t_0}(\mathbf{p}, \mathbf{x}_{\text{init}}, \mathbf{y}_{\text{init}}), \\ \mathbf{x}_{\text{init}} &= \mathbf{f}_0(\mathbf{p}), \end{aligned} \tag{4}$$

where $\mathbf{f}_{t_0} \equiv \mathbf{f}(t_0, \cdot, \cdot, \cdot)$ and $\mathbf{g}_{t_0} \equiv \mathbf{g}(t_0, \cdot, \cdot, \cdot)$. Equation (4) admits $n_p$ degrees of freedom for specification of the problem parameter $\mathbf{p}$ since it consists of $2n_x + n_y$ equations with $n_p + n_x + n_y + n_x$ variables ($\mathbf{p}$, $\mathbf{x}_{\text{init}}$, $\mathbf{y}_{\text{init}}$, and

$\dot{\mathbf{x}}_{\text{init}}$, respectively). The nonemptiness of $G_{\text{C},0}$ is a necessary and sufficient condition for the solvability of (4): if $(t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0) \in \mathcal{G}_{\text{C},0}$, then $(\mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0, \mathbf{f}_{t_0}(\mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0))$ is a solution of (4). On the other hand, if

$$(\mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0, \dot{\mathbf{x}}_0) \in \pi_{p,x,y}(D; t_0) \times \mathbb{R}^{n_x} \subset \mathbb{R}^{n_p} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_x}$$

is a solution of (4), $\mathbf{x}_0 = \mathbf{f}_0(\mathbf{p}_0)$ and $\mathbf{g}_{t_0}(\mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0) = \mathbf{0}_{n_y}$ must be satisfied. Moreover, given a parameter value and an associated solution of (4), the unique solvability of (4) in a neighborhood of said parameter is guaranteed under regularity.

**Assumption 4.12.** Suppose that there exists an open and connected set $N \subset D$ for which $t_0 \in \pi_t N$ and the mapping $\mathbf{f}_{t_0}$ is locally Lipschitz continuous on $\pi_{p,x,y}(N; t_0)$.

**Theorem 4.13.** Let Assumptions 4.1, 4.2, and 4.12 hold. If $(t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0) \in G_{\text{C},0} \cap G_{\text{R}} \cap N$, then there exist a neighborhood $N(\mathbf{p}_0) \subset \pi_p N$ of $\mathbf{p}_0$ and a Lipschitz continuous function $\mathbf{r}_0 : N(\mathbf{p}_0) \to \mathbb{R}^{n_y}$ such that, for each $\mathbf{p} \in N(\mathbf{p}_0)$,

$$(\mathbf{p}, \mathbf{f}_0(\mathbf{p}), \mathbf{r}_0(\mathbf{p}), \mathbf{f}_{t_0}(\mathbf{p}, \mathbf{f}_0(\mathbf{p}), \mathbf{r}_0(\mathbf{p})))$$

is the unique solution of (4) in a neighborhood of $(\mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0, \mathbf{f}_{t_0}(\mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0))$ and

$$\{(t, \mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y) : t = t_0, \mathbf{p} \in N(\mathbf{p}_0), \boldsymbol{\eta}_x = \mathbf{f}_0(\mathbf{p}), \boldsymbol{\eta}_y = \mathbf{r}_0(\mathbf{p})\} \subset G_{\text{C},0} \cap G_{\text{R}} \cap N.$$

*Proof.* Equation (4) is equivalently written as $\mathbf{F}(t, \mathbf{p}, \mathbf{x}_{\text{init}}, \mathbf{y}_{\text{init}}, \dot{\mathbf{x}}_{\text{init}}) = \mathbf{0}_{n_x+n_y+n_x+1}$ where

$$\mathbf{F} : D_{\mathbf{F}} \to \mathbb{R}^{n_x+n_y+n_x+1} : (t, \mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y, \boldsymbol{\eta}_{\dot{x}}) \mapsto \begin{bmatrix} \mathbf{f}_{t_0}(\mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y) - \boldsymbol{\eta}_{\dot{x}} \\ \mathbf{g}(t, \mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y) \\ \mathbf{f}_0(\mathbf{p}) - \boldsymbol{\eta}_x \\ t - t_0 \end{bmatrix},$$

and

$$D_{\mathbf{F}} := (D \cap (\mathbb{R} \times \pi_{p,x,y}(D; t_0))) \times \mathbb{R}^{n_x} \subset D \times \mathbb{R}^{n_x}$$

is open by Lemma 2.2. Let $\dot{\mathbf{x}}_0 := \mathbf{f}_{t_0}(\mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0)$ and $\boldsymbol{\omega}_0 := (t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0, \dot{\mathbf{x}}_0)$. Since $(t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0) \in G_{\text{C},0}$, $\mathbf{F}(\boldsymbol{\omega}_0) = \mathbf{0}_{n_x+n_y+n_x+1}$ by inspection. Let $Z_{\mathbf{F}} \subset D_{\mathbf{F}}$ denote the zero-measure subset on which $\mathbf{F}$ is not differentiable.

It is claimed that $\pi_{1,3,4,5}\partial\mathbf{F}(\boldsymbol{\omega}_0)$ is of maximal rank. If not, there exist $\mathbf{P}^* \in \mathbb{R}^{(n_x+n_y+n_x+1)\times n_p}$ and

$$[\mathbf{T}^* \quad \mathbf{X}^* \quad \mathbf{Y}^* \quad \dot{\mathbf{X}}^*] \in \pi_{1,3,4,5}\partial\mathbf{F}(\boldsymbol{\omega}_0)$$

such that $[\mathbf{T}^* \quad \mathbf{X}^* \quad \mathbf{Y}^* \quad \dot{\mathbf{X}}^*]$ is singular and

$$[\mathbf{T}^* \quad \mathbf{P}^* \quad \mathbf{X}^* \quad \mathbf{Y}^* \quad \dot{\mathbf{X}}^*] \in \partial\mathbf{F}(\boldsymbol{\omega}_0).$$

By definition of the Clarke Jacobian, there exist $k^* \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_{k^*} \geq 0$, and

$$[\mathbf{T}_1 \quad \mathbf{P}_1 \quad \mathbf{X}_1 \quad \mathbf{Y}_1 \quad \dot{\mathbf{X}}_1], \ldots, [\mathbf{T}_{k^*} \quad \mathbf{P}_{k^*} \quad \mathbf{X}_{k^*} \quad \mathbf{Y}_{k^*} \quad \dot{\mathbf{X}}_{k^*}] \in \partial_{\text{B}}\mathbf{F}(\boldsymbol{\omega}_0)$$

such that $\sum_{i=1}^{k^*} \lambda_i = 1$ and

$$[\mathbf{T}^* \quad \mathbf{P}^* \quad \mathbf{X}^* \quad \mathbf{Y}^* \quad \dot{\mathbf{X}}^*] = \sum_{i=1}^{k^*} \lambda_i [\mathbf{T}_i \quad \mathbf{P}_i \quad \mathbf{X}_i \quad \mathbf{Y}_i \quad \dot{\mathbf{X}}_i].$$

By definition of the B-subdifferential, for each $i \in \{1, \ldots, k^*\}$, there exists a sequence of vectors $\{\boldsymbol{\omega}_{(i_j)}\}$ such that

$$\boldsymbol{\omega}_{(i_j)} := (t_{(i_j)}, \mathbf{p}_{(i_j)}, \mathbf{x}_{(i_j)}, \mathbf{y}_{(i_j)}, \dot{\mathbf{x}}_{(i_j)}) \in D_{\mathbf{F}} \setminus Z_{\mathbf{F}}, \quad \forall j \in \mathbb{N},$$

$\lim_{j\to\infty}\boldsymbol{\omega}_{(i_j)} = (t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0, \dot{\mathbf{x}}_0)$, and

$$\begin{aligned}
&[\mathbf{T}_i \quad \mathbf{P}_i \quad \mathbf{X}_i \quad \mathbf{Y}_i \quad \dot{\mathbf{X}}_i] \\
&\quad = \lim_{j\to\infty}\left[\frac{\partial \mathbf{F}}{\partial t}(\boldsymbol{\omega}_{(i_j)}) \quad \frac{\partial \mathbf{F}}{\partial \mathbf{p}}(\boldsymbol{\omega}_{(i_j)}) \quad \frac{\partial \mathbf{F}}{\partial \mathbf{x}_{\mathrm{init}}}(\boldsymbol{\omega}_{(i_j)}) \quad \frac{\partial \mathbf{F}}{\partial \mathbf{y}_{\mathrm{init}}}(\boldsymbol{\omega}_{(i_j)}) \quad \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}_{\mathrm{init}}}(\boldsymbol{\omega}_{(i_j)})\right], \\
&\quad = \lim_{j\to\infty}\begin{bmatrix}
\mathbf{0}_{n_x} & \frac{\partial \mathbf{f}_{t_0}}{\partial \mathbf{p}}(\mathbf{p}_{(i_j)}, \mathbf{x}_{(i_j)}, \mathbf{y}_{(i_j)}) & \frac{\partial \mathbf{f}_{t_0}}{\partial \mathbf{x}_{\mathrm{init}}}(\mathbf{p}_{(i_j)}, \mathbf{x}_{(i_j)}, \mathbf{y}_{(i_j)}) & \frac{\partial \mathbf{f}_{t_0}}{\partial \mathbf{y}_{\mathrm{init}}}(\mathbf{p}_{(i_j)}, \mathbf{x}_{(i_j)}, \mathbf{y}_{(i_j)}) & -\mathbf{I}_{n_x} \\
\frac{\partial \mathbf{g}}{\partial t}(\boldsymbol{\gamma}_{(i_j)}) & \frac{\partial \mathbf{g}}{\partial \mathbf{p}}(\boldsymbol{\gamma}_{(i_j)}) & \frac{\partial \mathbf{g}}{\partial \mathbf{x}_{\mathrm{init}}}(\boldsymbol{\gamma}_{(i_j)}) & \frac{\partial \mathbf{g}}{\partial \mathbf{y}_{\mathrm{init}}}(\boldsymbol{\gamma}_{(i_j)}) & \mathbf{0}_{n_y\times n_x} \\
\mathbf{0}_{n_x} & \frac{\partial \mathbf{f}_0}{\partial \mathbf{p}}(\mathbf{p}_{(i_j)}) & -\mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} \\
1 & \mathbf{0}_{1\times n_p} & \mathbf{0}_{1\times n_x} & \mathbf{0}_{1\times n_y} & \mathbf{0}_{1\times n_x}
\end{bmatrix}, \\
&\quad =: \begin{bmatrix}
\mathbf{0}_{n_x} & \mathbf{P}_{i,1} & \mathbf{X}_{i,1} & \mathbf{Y}_{i,1} & -\mathbf{I}_{n_x} \\
\mathbf{T}_{i,2} & \mathbf{P}_{i,2} & \mathbf{X}_{i,2} & \mathbf{Y}_{i,2} & \mathbf{0}_{n_y\times n_x} \\
\mathbf{0}_{n_x} & \mathbf{P}_{i,3} & -\mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} \\
1 & \mathbf{0}_{1\times n_p} & \mathbf{0}_{1\times n_x} & \mathbf{0}_{1\times n_y} & \mathbf{0}_{1\times n_x}
\end{bmatrix},
\end{aligned}$$

where $\boldsymbol{\gamma}_{(i_j)} := (t_{(i_j)}, \mathbf{p}_{(i_j)}, \mathbf{x}_{(i_j)}, \mathbf{y}_{(i_j)})$ for all $j \in \mathbb{N}$.

Observe that $\sum_{i=1}^{k^*}\lambda_i \mathbf{Y}_{i,2}$ is singular since $[\mathbf{T}^* \quad \mathbf{X}^* \quad \mathbf{Y}^* \quad \dot{\mathbf{X}}^*]$ is singular and

$$[\mathbf{T}^* \quad \mathbf{X}^* \quad \mathbf{Y}^* \quad \dot{\mathbf{X}}^*] = \sum_{i=1}^{k^*}\lambda_i[\mathbf{T}_i \quad \mathbf{X}_i \quad \mathbf{Y}_i \quad \dot{\mathbf{X}}_i] = \begin{bmatrix}
\mathbf{0}_{n_x} & \sum_{i=1}^{k^*}\lambda_i \mathbf{X}_{i,1} & \sum_{i=1}^{k^*}\lambda_i \mathbf{Y}_{i,1} & -\mathbf{I}_{n_x} \\
\sum_{i=1}^{k^*}\lambda_i \mathbf{T}_{i,2} & \sum_{i=1}^{k^*}\lambda_i \mathbf{X}_{i,2} & \sum_{i=1}^{k^*}\lambda_i \mathbf{Y}_{i,2} & \mathbf{0}_{n_y\times n_x} \\
\mathbf{0}_{n_x} & -\mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} \\
1 & \mathbf{0}_{1\times n_x} & \mathbf{0}_{1\times n_y} & \mathbf{0}_{1\times n_x}
\end{bmatrix}.$$

It is also true that $\sum_{i=1}^{k^*}\lambda_i \mathbf{Y}_{i,2} \in \pi_4\partial\mathbf{g}(t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0)$ as

$$\begin{aligned}
\sum_{i=1}^{k^*}\lambda_i[\mathbf{T}_{i,2} \quad \mathbf{P}_{i,2} \quad \mathbf{X}_{i,2} \quad \mathbf{Y}_{i,2}] &\in \mathrm{conv}\left(\left\{\lim_{j\to\infty}\mathbf{Jg}(\boldsymbol{\eta}_{(j)}) : \lim_{j\to\infty}\boldsymbol{\eta}_{(j)} = (t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0), \quad \boldsymbol{\eta}_{(j)} \in D \setminus Z_{\mathbf{g}}, \forall j \in \mathbb{N}\right\}\right), \\
&= \partial\mathbf{g}(t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0),
\end{aligned}$$

since, for each $i \in \{1, \ldots, k^*\}$, $\boldsymbol{\omega}_{(i_j)} \in D_{\mathbf{F}} \setminus Z_{\mathbf{F}}$ for all $j \in \mathbb{N}$ implies that $\boldsymbol{\gamma}_{(i_j)} \in D \setminus Z_{\mathbf{g}}$ for all $j \in \mathbb{N}$, where $Z_{\mathbf{g}} \subset D$ is the zero-measure subset on which $\mathbf{g}$ is not differentiable. However, $\pi_4\partial\mathbf{g}(t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0)$ is of maximal rank since $(t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0) \in G_{\mathrm{R}}$. Therefore $\pi_{1,3,4,5}\partial\mathbf{F}(\boldsymbol{\omega}_0)$ is of maximal rank by contraposition.

Theorem 3.5 can therefore be applied to yield the following: there exist $\delta_1, \rho > 0$ and a Lipschitz continuous function

$$\boldsymbol{\phi}_0 \equiv (q_0, \mathbf{u}_0, \mathbf{r}_0, \mathbf{v}_0) : B_{\delta_1}(\mathbf{p}_0) \subset \pi_p N \to \mathbb{R}^{1+n_x+n_y+n_x}$$

such that $\pi_{1,3,4,5}\partial\mathbf{F}(t, \mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y, \boldsymbol{\eta}_{\dot{x}})$ is of maximal rank for all $(t, \mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y, \boldsymbol{\eta}_{\dot{x}}) \in B_\rho(\boldsymbol{\omega}_0) \subset N$ and, for each $\mathbf{p} \in B_{\delta_1}(\mathbf{p}_0)$,

$$(q_0(\mathbf{p}), \mathbf{p}, \mathbf{u}_0(\mathbf{p}), \mathbf{r}_0(\mathbf{p}), \mathbf{v}_0(\mathbf{p}))$$

is the unique vector in $B_\rho(\boldsymbol{\omega}_0)$ satisfying

$$\mathbf{F}(q_0(\mathbf{p}), \mathbf{p}, \mathbf{u}_0(\mathbf{p}), \mathbf{r}_0(\mathbf{p}), \mathbf{v}_0(\mathbf{p})) = \mathbf{0}_{n_x+n_y+n_x+1}.$$

By inspection, $q_0(\mathbf{p}) = t_0$, $\mathbf{u}_0(\mathbf{p}) = \mathbf{f}_0(\mathbf{p})$ and $\mathbf{v}_0(\mathbf{p}) = \mathbf{f}_{t_0}(\mathbf{p}, \mathbf{f}_0(\mathbf{p}), \mathbf{r}_0(\mathbf{p}))$ for all $\mathbf{p} \in B_{\delta_1}(\mathbf{p}_0)$. Therefore, for each $\mathbf{p} \in B_{\delta_1}(\mathbf{p}_0)$, $(t_0, \mathbf{p}, \mathbf{f}_0(\mathbf{p}), \mathbf{r}_0(\mathbf{p}), \mathbf{f}_{t_0}(\mathbf{p}, \mathbf{f}_0(\mathbf{p}), \mathbf{r}_0(\mathbf{p})))$ is the unique vector in $B_\rho(\boldsymbol{\omega}_0)$ satisfying (4). It follows from the discussion preceding this theorem that

$$\{(t_0, \mathbf{p}, \mathbf{f}_0(\mathbf{p}), \mathbf{r}_0(\mathbf{p})) : \mathbf{p} \in B_{\delta_1}(\mathbf{p}_0)\} \subset G_{\mathrm{C},0}. \tag{5}$$

By maximal rankness of $\pi_4\partial\mathbf{g}(t_0,\mathbf{p}_0,\mathbf{x}_0,\mathbf{y}_0)$, there exists $\alpha > 0$ such that $\pi_4\partial\mathbf{g}(t,\mathbf{p},\boldsymbol{\eta}_x,\boldsymbol{\eta}_y)$ is of maximal rank for all $(t,\mathbf{p},\boldsymbol{\eta}_x,\boldsymbol{\eta}_y) \in B_\alpha(t_0,\mathbf{p}_0,\mathbf{x}_0,\mathbf{y}_0) \subset N$ by Lemma 3.3. Since $B_\alpha(t_0,\mathbf{p}_0,\mathbf{x}_0,\mathbf{y}_0)$ is open and the mapping $\mathbf{p} \mapsto (t_0,\mathbf{p},\mathbf{f}_0(\mathbf{p}),\mathbf{r}_0(\mathbf{p}))$ is continuous on $B_{\delta_1}(\mathbf{p}_0)$, there exists $\delta \in (0,\delta_1]$ such that $(t_0,\mathbf{p},\mathbf{f}_0(\mathbf{p}),\mathbf{r}_0(\mathbf{p})) \in B_\alpha(t_0,\mathbf{p}_0,\mathbf{x}_0,\mathbf{y}_0)$ for all $\mathbf{p} \in B_\delta(\mathbf{p}_0)$. Hence,

$$\{(t_0,\mathbf{p},\mathbf{f}_0(\mathbf{p}),\mathbf{r}_0(\mathbf{p})) : \mathbf{p} \in B_\delta(\mathbf{p}_0)\} \subset G_\mathrm{R} \cap N. \tag{6}$$

The result holds with $N(\mathbf{p}_0) := B_\delta(\mathbf{p}_0)$ by virtue of (5) and (6). □

**Example 4.14.** Consider (3) with $t_0 := 0$ at initialization:

$$\begin{aligned} \dot{x}_\mathrm{init} &= -1, \\ 0 &= |x_\mathrm{init}| + |y_\mathrm{init}| - 1, \\ x_\mathrm{init} &= \min\{0,p\}. \end{aligned} \tag{7}$$

Assumptions 4.1, 4.2, and 4.12 hold with $N := \mathbb{R}^4$. In this case,

$$G_{\mathrm{C},0} \cap G_\mathrm{R} = \{(t,p,\eta_x,\eta_y) : t = t_0, \eta_x = \min\{0,p\}, \eta_y \neq 0, |\eta_x| + |\eta_y| = 1\}.$$

Let $(p_0,x_0,y_0) := (0,0,1)$. Then $(t_0,p_0,x_0,y_0) \in G_{\mathrm{C},0} \cap G_\mathrm{R} \cap N$. For each $p \in N(p_0) := (-0.5,0.5)$,

$$(p, x_\mathrm{init}(p), y_\mathrm{init}(p), \dot{x}_\mathrm{init}(p)) \equiv \begin{cases} (p,p,1-|p|,-1), & \text{if } p \in (-0.5,0], \\ (0,0,1,-1), & \text{if } p \in (0,0.5), \end{cases}$$

is the unique solution of (7) in $B_{0.5\sqrt{3}}(0,0,1,-1)$, in accordance with Theorem 4.13. The Lipschitz continuous function

$$r_0 : N(p_0) \to \mathbb{R} : p \mapsto \begin{cases} 1-|p|, & \text{if } p \in (-0.5,0], \\ 1, & \text{if } p \in (0,0.5), \end{cases}$$

satisfies

$$\{(t,p,\eta_x,\eta_y) : t = t_0, p \in N(p_0), \eta_x = \min\{0,p\}, \eta_y = r_0(p)\} \subset G_{\mathrm{C},0} \cap G_\mathrm{R} \cap N.$$

Highlighted in the next example, Theorem 4.13 is sufficient but not necessary.

**Example 4.15.** Consider (3) with $t_0 := 1$ at initialization:

$$\begin{aligned} \dot{x}_\mathrm{init} &= 1.5|1 - y_\mathrm{init}|^{\frac{1}{3}}, \\ 0 &= |x_\mathrm{init}| + |y_\mathrm{init}| - 1, \\ x_\mathrm{init} &= \min\{0,p\}. \end{aligned} \tag{8}$$

Assumptions 4.1, 4.2, and 4.12 hold with $N := \{(t,p,\eta_x,\eta_y) \in \mathbb{R}^4 : \eta_y < 1\}$. Let $(p_0,x_0,y_0) := (0,0,1)$. Then $(t_0,p_0,x_0,y_0) \in G_{\mathrm{C},0} \cap G_\mathrm{R}$ but $(t_0,p_0,x_0,y_0) \notin N$. However, for each $p \in N(p_0) := (-0.5,0.5)$,

$$(p, x_\mathrm{init}(p), y_\mathrm{init}(p), \dot{x}_\mathrm{init}(p)) \equiv \begin{cases} (p,p,1-|p|,1.5|p|^{\frac{1}{3}}), & \text{if } p \in (-0.5,0], \\ (0,0,1,0), & \text{if } p \in (0,0.5], \end{cases}$$

is the unique solution of (8) in $B_{0.5\sqrt{3}}(0,0,1,0)$.

**Remark 4.16.** There is no solution of (3) on $T \times \{-2\}$ for any connected set $T \subset \mathbb{R}$; a consequence of inconsistent initialization for the parameter $p := -2$ because $G_{\mathrm{C},0} \cap \{(t,p,\eta_x,\eta_y) : p = -2\} = \emptyset$.

If $\mathbf{f}$ is $C^1$ on $D$, $\mathbf{g}$ is $C^1$ on $D$, and $(t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0) \in G_{\mathrm{C},0} \cap G_{\mathrm{R}}$, then there exist a neighborhood $N(t_0)$ of $t_0$ and a regular $C^1$-solution of (2) through $\{(t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0)\}$ on $N(t_0) \times \{\mathbf{p}_0\}$ (see Theorems 4.13 and 4.18 in [1]). Here, existence of a solution of (2) is demonstrated under regularity and consistency of initial data, local Lipschitz continuity of $\mathbf{g}$, and the following Carathéodory existence conditions on $\mathbf{f}$ (see, e.g., [53, 54]).

**Assumption 4.17.** Suppose that there exists an open and connected set $N \subset D$ for which the following Carathéodory conditions hold:

(i) the mapping $\mathbf{f}(\cdot, \mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y) : \pi_t(N; (\mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y)) \to \mathbb{R}^{n_x}$ is measurable on its domain for each $(\mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y) \in \pi_{p,x,y}N$;

(ii) the mapping $\mathbf{f}(t, \cdot, \cdot, \cdot) : \pi_{p,x,y}(N; t) \to \mathbb{R}^{n_x}$ is continuous on its domain for each $t \in \pi_t N \setminus Z_{\mathbf{f}}$, where $Z_{\mathbf{f}}$ is a zero-measure subset;

(iii) there exists a Lebesgue integrable function $m_{\mathbf{f}} : \pi_t N \to \mathbb{R}_+ \cup \{+\infty\}$ such that

$$\|\mathbf{f}(t, \mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y)\| \leq m_{\mathbf{f}}(t), \quad \forall (t, \mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y) \in N.$$

**Remark 4.18.** The composition of a (Lebesgue) measurable function with a measurable function need not be measurable. Though the reverse is true, even the composition of a measurable function with a continuous function need not be measurable (see, e.g., Page 241 in [55]); the possible nonmeasurability stems from the fact that the preimage of a measurable set under a continuous mapping need not be measurable. However, a composition of a function satisfying the Carathéodory conditions with a measurable function on a connected and compact set is Lebesgue integrable, and therefore measurable (see Lemma 1 in Chapter 1, Section 1 [53], which originally appears in [56]).

By completeness of the Lebesgue measure, the almost everywhere pointwise limit of a measurable function is a measurable function (see, e.g., Proposition 21 in Chapter 3, Section 5 [57]). Consequently, a composition of a function satisfying the Carathéodory conditions with a measurable function is measurable (see, for example, Lemma 8.2.3 in [58]). This result is restated in the following useful form.

**Lemma 4.19.** Let Assumption 4.17 hold. Let $\mathbf{h} : T \subset \mathbb{R} \to \pi_{p,x,y}N$ be measurable on $T$ and satisfy $\{(t, \mathbf{h}(t)) : t \in T\} \subset N$. Then the mapping $t \mapsto \mathbf{f}(t, \mathbf{h}(t))$ is measurable on $T$.

Sufficient conditions for local existence of regular solutions are now given.

**Theorem 4.20.** Let Assumptions 4.2 and 4.17 hold. Suppose that $(t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0) \in G_{\mathrm{C},0} \cap G_{\mathrm{R}} \cap N$. Then there exist $\alpha > 0$ and a regular solution of (2) on $[t_0 - \alpha, t_0 + \alpha] \times \{\mathbf{p}_0\}$ through $\{(t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0)\}$ in $N$.

*Proof.* Since $(t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0) \in G_{\mathrm{C},0} \cap G_{\mathrm{R}} \cap N$ and $\mathbf{g}$ is locally Lipschitz continuous on $N$, Theorem 3.5 implies the existence of $\delta, \rho > 0$ and a Lipschitz continuous function $\mathbf{r} : B_\delta(t_0, \mathbf{p}_0, \mathbf{x}_0) \subset \pi_{t,p,x}N \to \mathbb{R}^{n_y}$ such that, for each $(t, \mathbf{p}, \boldsymbol{\eta}_x) \in B_\delta(t_0, \mathbf{p}_0, \mathbf{x}_0)$, $(t, \mathbf{p}, \boldsymbol{\eta}_x, \mathbf{r}(t, \mathbf{p}, \boldsymbol{\eta}_x))$ is the unique vector in $B_\rho(t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0) \subset N$ satisfying $\mathbf{0}_{n_y} = \mathbf{g}(t, \mathbf{p}, \boldsymbol{\eta}_x, \mathbf{r}(t, \mathbf{p}, \boldsymbol{\eta}_x))$, and $\pi_4 \partial \mathbf{g}(t, \mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y)$ is of maximal rank for all $(t, \mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y) \in B_\rho(t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0)$.

Define the mappings

$$\begin{aligned} \mathbf{q} &: B_\delta(t_0, \mathbf{p}_0, \mathbf{x}_0) \to \pi_{p,x,y}N : (t, \mathbf{p}, \boldsymbol{\eta}_x) \mapsto (\mathbf{p}, \boldsymbol{\eta}_x, \mathbf{r}(t, \mathbf{p}, \boldsymbol{\eta}_x)), \\ \bar{\mathbf{f}} &: B_\delta(t_0, \mathbf{p}_0, \mathbf{x}_0) \to \mathbb{R}^{n_p + n_x} : (t, \mathbf{p}, \boldsymbol{\eta}_x) \mapsto \begin{bmatrix} \mathbf{0}_{n_p} \\ \mathbf{f}(t, \mathbf{q}(t, \mathbf{p}, \boldsymbol{\eta}_x)) \end{bmatrix}. \end{aligned}$$

Then

$$\{(t, \mathbf{q}(t, \mathbf{p}, \boldsymbol{\eta}_x)) : (t, \mathbf{p}, \boldsymbol{\eta}_x) \in B_\delta(t_0, \mathbf{p}_0, \mathbf{x}_0)\} \subset B_\rho(t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0) \subset N. \tag{9}$$

Since $B_\delta(t_0, \mathbf{p}_0, \mathbf{x}_0)$ is open, it is possible to choose $a^*, b^* > 0$ such that

$$W := \{(t, \mathbf{p}, \boldsymbol{\eta}_x) \in \mathbb{R} \times \mathbb{R}^{n_p} \times \mathbb{R}^{n_x} : |t - t_0| \leq a^*, \mathbf{p} = \mathbf{p}_0, \|\boldsymbol{\eta}_x - \mathbf{x}_0\| \leq b^*\} \subset B_\delta(t_0, \mathbf{p}_0, \mathbf{x}_0).$$

For each $(t, \mathbf{p}, \boldsymbol{\eta}_x) \in W$,

$$\|\bar{\mathbf{f}}(t, \mathbf{p}, \boldsymbol{\eta}_x)\| = \|\mathbf{f}(t, \mathbf{p}, \boldsymbol{\eta}_x, \mathbf{r}(t, \mathbf{p}, \boldsymbol{\eta}_x))\| \leq m_{\mathbf{f}}(t),$$

by (9). For each $t \in \pi_t W \setminus Z_{\mathbf{f}} = [t_0 - a^*, t_0 + a^*] \setminus Z_{\mathbf{f}}$, the mapping

$$\bar{\mathbf{f}}(t, \cdot, \cdot) \equiv \begin{bmatrix} \mathbf{0}_{n_p} \\ \mathbf{f}(t, \cdot, \cdot, \mathbf{r}(t, \cdot, \cdot)) \end{bmatrix} : \pi_{p,x}(B_\delta(t_0, \mathbf{p}_0, \mathbf{x}_0); t) \subset \pi_{p,x} N \to \mathbb{R}^{n_p + n_x}$$

is continuous on $\pi_{p,x}(B_\delta(t_0, \mathbf{p}_0, \mathbf{x}_0); t) \supset \pi_{p,x} W$ by continuity of $\mathbf{r}$ on $B_\delta(t_0, \mathbf{p}_0, \mathbf{x}_0)$, (9), and the fact that $\mathbf{f}(t, \cdot, \cdot, \cdot)$ is continuous on $\pi_{p,x,y}(N; t)$ for each $t \in \pi_t N \setminus Z_{\mathbf{f}} \supset [t_0 - a^*, t_0 + a^*] \setminus Z_{\mathbf{f}}$. For each $(\mathbf{p}, \boldsymbol{\eta}_x) \in \pi_{p,x} W$, the mapping

$$\bar{\mathbf{f}}(\cdot, \mathbf{p}, \boldsymbol{\eta}_x) \equiv \begin{bmatrix} \mathbf{0}_{n_p} \\ \mathbf{f}(\cdot, \mathbf{p}, \boldsymbol{\eta}_x, \mathbf{r}(\cdot, \mathbf{p}, \boldsymbol{\eta}_x)) \end{bmatrix} : \pi_t(B_\delta(t_0, \mathbf{p}_0, \mathbf{x}_0); (\mathbf{p}, \boldsymbol{\eta}_x)) \subset \pi_t N \to \mathbb{R}^{n_p + n_x}$$

is measurable on $[t_0 - a^*, t_0 + a^*] \subset \pi_t(B_\delta(t_0, \mathbf{p}_0, \mathbf{x}_0); (\mathbf{p}, \boldsymbol{\eta}_x))$ by Lemma 4.19 since the mapping $t \mapsto (\mathbf{p}, \boldsymbol{\eta}_x, \mathbf{r}(t, \mathbf{p}, \boldsymbol{\eta}_x))$ is continuous (and thus measurable) on the open (and therefore measurable) set $\pi_t(B_\delta(t_0, \mathbf{p}_0, \mathbf{x}_0); (\mathbf{p}, \boldsymbol{\eta}_x)) \supset [t_0 - a^*, t_0 + a^*]$ and $\mathbf{f}$ satisfies the Carathéodory existence conditions on

$$N \supset B_\rho(t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0) \supset \{(t, \mathbf{p}, \boldsymbol{\eta}_x, \mathbf{r}(t, \mathbf{p}, \boldsymbol{\eta}_x)) : t \in [t_0 - a^*, t_0 + a^*]\}$$

by assumption.

Letting $\mathbf{c}_0 := (\mathbf{p}_0, \mathbf{x}_0)$, the right-hand side of the ODE system

$$\begin{aligned} \dot{\mathbf{u}}(t) &= \bar{\mathbf{f}}(t, \mathbf{u}(t)), \\ \mathbf{u}(t_0) &= \mathbf{c}_0, \end{aligned} \tag{10}$$

satisfies the Carathéodory existence conditions on $W$ by the above arguments. By Theorem 1 in Chapter 1, Section 1 [53], there exist $\alpha \in (0, a^*]$ and an absolutely continuous mapping $\mathbf{u}$ defined on $[t_0 - \alpha, t_0 + \alpha]$ such that

$$\mathbf{u}(t) = \mathbf{c}_0 + \int_{t_0}^{t} \bar{\mathbf{f}}(s, \mathbf{u}(s)) \mathrm{d}s, \quad \forall t \in [t_0 - \alpha, t_0 + \alpha],$$

and $\{(t, \mathbf{u}(t)) : t \in [t_0 - \alpha, t_0 + \alpha]\} \subset W$. Let

$$(\boldsymbol{\rho}, \mathbf{x}) : [t_0 - \alpha, t_0 + \alpha] \times \{\mathbf{p}_0\} \to \pi_{p,x} W : (t, \mathbf{p}) \mapsto \mathbf{u}(t).$$

By inspection, $\boldsymbol{\rho}(t, \mathbf{p}_0) = \mathbf{p}_0$ for all $t \in [t_0 - \alpha, t_0 + \alpha]$ and the mapping $\mathbf{x}(\cdot, \mathbf{p}_0)$ satisfies

$$\mathbf{x}(t, \mathbf{p}_0) = \mathbf{f}_0(\mathbf{p}_0) + \int_{t_0}^{t} \mathbf{f}(s, \mathbf{p}_0, \mathbf{x}(s, \mathbf{p}_0), \mathbf{r}(s, \mathbf{p}_0, \mathbf{x}(s, \mathbf{p}_0))) \mathrm{d}s, \quad \forall t \in [t_0 - \alpha, t_0 + \alpha],$$

and is therefore absolutely continuous by Lebesgue integrability of $m_{\mathbf{f}}$ on $\pi_t N \supset [t_0 - \alpha, t_0 + \alpha]$.

Define the mapping

$$\mathbf{y} : [t_0 - \alpha, t_0 + \alpha] \times \{\mathbf{p}_0\} \to \mathbb{R}^{n_y} : (t, \mathbf{p}) \mapsto \mathbf{r}(t, \mathbf{p}, \mathbf{x}(t, \mathbf{p})).$$

Observe that

$$B_\delta(t_0, \mathbf{p}_0, \mathbf{x}_0) \supset W \supset \{(t, \mathbf{u}(t)) : t \in [t_0 - \alpha, t_0 + \alpha]\} = \{(t, \mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0)) : t \in [t_0 - \alpha, t_0 + \alpha]\}$$

implies that

$$\begin{aligned} \{(t, \mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0), \mathbf{y}(t, \mathbf{p}_0)) : t \in [t_0 - \alpha, t_0 + \alpha]\} &= \{(t, \mathbf{q}(t, \mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0))) : t \in [t_0 - \alpha, t_0 + \alpha]\}, \\ &\subset B_\rho(t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0), \\ &\subset N. \end{aligned}$$

Since $\pi_4 \partial \mathbf{g}(t, \mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y)$ is of maximal rank for all $(t, \mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y) \in B_\rho(t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0)$,

$$\{(t, \mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0), \mathbf{y}(t, \mathbf{p}_0)) : t \in [t_0 - \alpha, t_0 + \alpha]\} \subset G_{\mathrm{R}}.$$

By absolute continuity of $(\boldsymbol{\rho}(\cdot, \mathbf{p}_0), \mathbf{x}(\cdot, \mathbf{p}_0))$ on $[t_0 - \alpha, t_0 + \alpha]$ and Lipschitz continuity of $\mathbf{r}$ on $B_\delta(t_0, \mathbf{p}_0, \mathbf{x}_0)$, it follows from Lemma 2.5 that $\mathbf{y}(\cdot, \mathbf{p}_0) \equiv \mathbf{r} \circ (\cdot, \boldsymbol{\rho}(\cdot, \mathbf{p}_0), \mathbf{x}(\cdot, \mathbf{p}_0))$ is an absolutely continuous mapping on $[t_0 - \alpha, t_0 + \alpha]$. Moreover,

$$\mathbf{0}_{n_y} = \mathbf{g}(t, \mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0), \mathbf{r}(t, \mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0))) = \mathbf{g}(t, \mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0), \mathbf{y}(t, \mathbf{p}_0)), \quad \forall t \in [t_0 - \alpha, t_0 + \alpha]. \tag{11}$$

Hence, $(\mathbf{x}, \mathbf{y})$ is a regular solution of (2) on $[t_0 - \alpha, t_0 + \alpha] \times \{\mathbf{p}_0\}$ through $\{(t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0)\}$ in $N$. □

Given a regular solution, its uniqueness is ascertained under an analogous Carathéodory-style uniqueness condition (see, e.g., Theorem 2 in Chapter 1, Section 1 [53]) for nonsmooth DAEs.

**Assumption 4.21.** Suppose that there exist an open and connected set $N \subset D$ and a Lebesgue integrable function $k_{\mathbf{f}} : \pi_t N \to \mathbb{R}_+ \cup \{+\infty\}$ such that, for any $t \in \pi_t N$ and any $(\mathbf{p}_1, \boldsymbol{\eta}_{x_1}, \boldsymbol{\eta}_{y_1}), (\mathbf{p}_2, \boldsymbol{\eta}_{x_2}, \boldsymbol{\eta}_{y_2}) \in \pi_{p,x,y}(N; t)$,

$$\|\mathbf{f}(t, \mathbf{p}_1, \boldsymbol{\eta}_{x_1}, \boldsymbol{\eta}_{y_1}) - \mathbf{f}(t, \mathbf{p}_2, \boldsymbol{\eta}_{x_2}, \boldsymbol{\eta}_{y_2})\| \leq k_{\mathbf{f}}(t)\|(\mathbf{p}_1, \boldsymbol{\eta}_{x_1}, \boldsymbol{\eta}_{y_1}) - (\mathbf{p}_2, \boldsymbol{\eta}_{x_2}, \boldsymbol{\eta}_{y_2})\|.$$

**Theorem 4.22.** Let Assumptions 4.2 and 4.21 hold. Suppose that $\mathbf{z}$ is a regular solution of (2) on $T \times P$ through $\Omega_0$ in $N$. Then $\mathbf{z}$ is a unique regular solution of (2) on $T \times P$ through $\Omega_0$ in $N$.

*Proof.* Let $\widetilde{\mathbf{z}} \equiv (\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})$ be another solution of (2) on $\widetilde{T} \times \widetilde{P}$ through $\widetilde{\Omega}_0$ such that $T \cap \widetilde{T} \neq \{t_0\}$, $P \cap \widetilde{P} \neq \emptyset$, and

$$\{(t_0, \mathbf{p}, \mathbf{z}(t_0, \mathbf{p})) : \mathbf{p} \in P \cap \widetilde{P}\} = \{(t_0, \mathbf{p}, \widetilde{\mathbf{z}}(t_0, \mathbf{p})) : \mathbf{p} \in P \cap \widetilde{P}\}.$$

Suppose, without loss of generality, that $T \cap \widetilde{T} \cap [t_0, +\infty) \neq \emptyset$ and $\mathbf{z}(t, \mathbf{p}^*) \neq \widetilde{\mathbf{z}}(t, \mathbf{p}^*)$ for some $t \in T \cap \widetilde{T} \cap [t_0, +\infty)$ and some $\mathbf{p}^* \in P \cap \widetilde{P}$. Define the set

$$H := \{t \in T \cap \widetilde{T} \cap [t_0, +\infty) : \|\mathbf{z}(t, \mathbf{p}^*) - \widetilde{\mathbf{z}}(t, \mathbf{p}^*)\| > 0\},$$

which is nonempty and bounded below. Hence, $t^* := \inf H$ exists. By continuity of $\mathbf{z}(\cdot, \mathbf{p}^*)$ and $\widetilde{\mathbf{z}}(\cdot, \mathbf{p}^*)$, $\mathbf{z}(t^*, \mathbf{p}^*) = \widetilde{\mathbf{z}}(t^*, \mathbf{p}^*)$. By consistency,

$$\mathbf{0}_{n_y} = \mathbf{g}(t^*, \mathbf{p}^*, \mathbf{x}(t^*, \mathbf{p}^*), \mathbf{y}(t^*, \mathbf{p}^*)) = \mathbf{g}(t^*, \mathbf{p}^*, \widetilde{\mathbf{x}}(t^*, \mathbf{p}^*), \widetilde{\mathbf{y}}(t^*, \mathbf{p}^*)).$$

By regularity,

$$\pi_4 \partial \mathbf{g}(t^*, \mathbf{p}^*, \mathbf{x}(t^*, \mathbf{p}^*), \mathbf{y}(t^*, \mathbf{p}^*)) = \pi_4 \partial \mathbf{g}(t^*, \mathbf{p}^*, \widetilde{\mathbf{x}}(t^*, \mathbf{p}^*), \widetilde{\mathbf{y}}(t^*, \mathbf{p}^*))$$

is of maximal rank.

Let $\mathbf{z}^* := \mathbf{z}(t^*, \mathbf{p}^*)$, $\mathbf{x}^* := \mathbf{x}(t^*, \mathbf{p}^*)$, $\mathbf{y}^* := \mathbf{y}(t^*, \mathbf{p}^*)$. Note that $(t^*, \mathbf{p}^*, \mathbf{x}^*, \mathbf{y}^*) \in N$ since $\{(t, \mathbf{p}, \mathbf{x}(t, \mathbf{p}), \mathbf{y}(t, \mathbf{p})) : (t, \mathbf{p}) \in T \times P\} \subset N$. Theorem 3.5 implies the existence of $\delta, \rho > 0$ and a Lipschitz continuous function $\mathbf{r} : B_\delta(t^*, \mathbf{p}^*, \mathbf{x}^*) \subset \pi_{t,p,x} N \to \mathbb{R}^{n_y}$ such that for all $(t, \mathbf{p}, \boldsymbol{\eta}_x) \in B_\delta(t^*, \mathbf{p}^*, \mathbf{x}^*)$, $(t, \mathbf{p}, \boldsymbol{\eta}_x, \mathbf{r}(t, \mathbf{p}, \boldsymbol{\eta}_x))$ is the unique vector in $B_\rho(t^*, \mathbf{p}^*, \mathbf{x}^*, \mathbf{y}^*) \subset N$ which satisfies $\mathbf{0}_{n_y} = \mathbf{g}(t, \mathbf{p}, \boldsymbol{\eta}_x, \mathbf{r}(t, \mathbf{p}, \boldsymbol{\eta}_x))$. By continuity of $\mathbf{z}(\cdot, \mathbf{p}^*)$ and $\bar{\mathbf{z}}(\cdot, \mathbf{p}^*)$, there exists $\gamma > 0$ such that

$$\{(t, \mathbf{p}^*, \mathbf{x}(t, \mathbf{p}^*), \mathbf{y}(t, \mathbf{p}^*)) : t \in (t^* - \gamma, t^* + \gamma) \cap T \cap \widetilde{T}\} \subset B_\rho(t^*, \mathbf{p}^*, \mathbf{x}^*, \mathbf{y}^*)$$

and

$$\{(t, \mathbf{p}^*, \widetilde{\mathbf{x}}(t, \mathbf{p}^*), \widetilde{\mathbf{y}}(t, \mathbf{p}^*)) : t \in (t^* - \gamma, t^* + \gamma) \cap T \cap \widetilde{T}\} \subset B_\rho(t^*, \mathbf{p}^*, \mathbf{x}^*, \mathbf{y}^*).$$

Let $\hat{T} := (t^* - \gamma, t^* + \gamma) \cap T \cap \widetilde{T} \cap [t_0, +\infty)$. It is claimed that

$$H^* := \{t \in \hat{T} : \|\mathbf{x}(t, \mathbf{p}^*) - \widetilde{\mathbf{x}}(t, \mathbf{p}^*)\| > 0\}$$

is nonempty. Otherwise,

$$\mathbf{y}(t, \mathbf{p}^*) = \mathbf{r}(t, \mathbf{p}^*, \mathbf{x}(t, \mathbf{p}^*)) = \mathbf{r}(t, \mathbf{p}^*, \widetilde{\mathbf{x}}(t, \mathbf{p}^*)) = \widetilde{\mathbf{y}}(t, \mathbf{p}^*), \quad \forall t \in \hat{T}.$$

This implies that $\mathbf{z}(t, \mathbf{p}^*) = \widetilde{\mathbf{z}}(t, \mathbf{p}^*)$ for all $t \in \hat{T}$, which contradicts the definition of $t^*$. Hence, $H^*$ must be nonempty.

By definition, the mappings $\mathbf{x}(\cdot, \mathbf{p}^*)$ and $\widetilde{\mathbf{x}}(\cdot, \mathbf{p}^*)$ satisfy

$$\dot{\mathbf{x}}(t, \mathbf{p}^*) = \mathbf{f}(t, \mathbf{p}^*, \mathbf{x}(t, \mathbf{p}^*), \mathbf{r}(t, \mathbf{p}^*, \mathbf{x}(t, \mathbf{p}^*))), \quad \text{a.e. } t \in \hat{T}$$

and

$$\dot{\widetilde{\mathbf{x}}}(t, \mathbf{p}^*) = \mathbf{f}(t, \mathbf{p}^*, \widetilde{\mathbf{x}}(t, \mathbf{p}^*), \mathbf{r}(t, \mathbf{p}^*, \widetilde{\mathbf{x}}(t, \mathbf{p}^*))), \quad \text{a.e. } t \in \hat{T},$$

respectively. We proceed as in the proof of Theorem 2 in Chapter 1, Section 1 [53]: define the mapping

$$\psi : \hat{T} \to \mathbb{R}_+ : t \mapsto \|\mathbf{x}(t, \mathbf{p}^*) - \widetilde{\mathbf{x}}(t, \mathbf{p}^*)\|^2.$$

Then, almost everywhere on $\hat{T}$,

$$\dot{\psi}(t) = (\mathbf{f}(t, \mathbf{p}^*, \mathbf{x}(t, \mathbf{p}^*), \mathbf{r}(t, \mathbf{p}^*, \mathbf{x}(t, \mathbf{p}^*))) - \mathbf{f}(t, \mathbf{p}^*, \widetilde{\mathbf{x}}(t, \mathbf{p}^*), \mathbf{r}(t, \mathbf{p}^*, \widetilde{\mathbf{x}}(t, \mathbf{p}^*))))^{\mathrm{T}}(\mathbf{x}(t, \mathbf{p}^*) - \widetilde{\mathbf{x}}(t, \mathbf{p}^*)).$$

Since $\mathbf{r}$ is Lipschitz continuous on $B_\delta(t^*, \mathbf{p}^*, \mathbf{x}^*)$, there exists $k_\mathbf{r} \geq 0$ for which

$$\|\mathbf{r}(t_1, \mathbf{p}_1, \boldsymbol{\eta}_{x_1}) - \mathbf{r}(t_2, \mathbf{p}_2, \boldsymbol{\eta}_{x_2})\| \leq k_\mathbf{r}\|(t_1, \mathbf{p}_1, \boldsymbol{\eta}_{x_1}) - (t_2, \mathbf{p}_2, \boldsymbol{\eta}_{x_2})\|, \ \forall (t_1, \mathbf{p}_1, \boldsymbol{\eta}_{x_1}), (t_2, \mathbf{p}_2, \boldsymbol{\eta}_{x_2}) \in B_\delta(t^*, \mathbf{p}^*, \mathbf{x}^*).$$

Then, by Assumption 4.21 and the Cauchy-Schwarz inequality,

$$\begin{aligned}
&(\mathbf{f}(t, \mathbf{p}^*, \mathbf{x}(t, \mathbf{p}^*), \mathbf{r}(t, \mathbf{p}^*, \mathbf{x}(t, \mathbf{p}^*))) - \mathbf{f}(t, \mathbf{p}^*, \widetilde{\mathbf{x}}(t, \mathbf{p}^*), \mathbf{r}(t, \mathbf{p}^*, \widetilde{\mathbf{x}}(t, \mathbf{p}^*))))^{\mathrm{T}}(\mathbf{x}(t, \mathbf{p}^*) - \widetilde{\mathbf{x}}(t, \mathbf{p}^*)) \\
&\leq \|\mathbf{f}(t, \mathbf{p}^*, \mathbf{x}(t, \mathbf{p}^*), \mathbf{r}(t, \mathbf{p}^*, \mathbf{x}(t, \mathbf{p}^*))) - \mathbf{f}(t, \mathbf{p}^*, \widetilde{\mathbf{x}}(t, \mathbf{p}^*), \mathbf{r}(t, \mathbf{p}^*, \widetilde{\mathbf{x}}(t, \mathbf{p}^*)))\|\|\mathbf{x}(t, \mathbf{p}^*) - \widetilde{\mathbf{x}}(t, \mathbf{p}^*)\|, \\
&\leq k_\mathbf{f}(t)(1 + k_\mathbf{r})\|\mathbf{x}(t, \mathbf{p}^*) - \widetilde{\mathbf{x}}(t, \mathbf{p}^*)\|^2,
\end{aligned}$$

for all $t \in \hat{T}$. Hence,

$$\dot{\psi}(t) \leq k_\mathbf{f}(t)(1 + k_\mathbf{r})\|\mathbf{x}(t, \mathbf{p}^*) - \widetilde{\mathbf{x}}(t, \mathbf{p}^*)\|^2 = k_\mathbf{f}(t)(1 + k_\mathbf{r})\psi(t), \quad \text{a.e. } t \in \hat{T},$$

implying that $\dot{\psi}(t) - k_\mathbf{f}(t)(1 + k_\mathbf{r})\psi(t) \leq 0$ almost everywhere on $\hat{T}$. Thus, the absolutely continuous mapping

$$t \mapsto \psi(t) \exp\left(-(1 + k_\mathbf{r}) \int_{t_0}^{t} k_\mathbf{f}(s) \mathrm{d}s\right)$$

is nonincreasing almost everywhere on $\hat{T}$. Since $\psi(t_0) = 0$, $\psi(t) = 0$ for all $t \in \hat{T}$. This implies that $H^*$ is empty, which is a contradiction. □

**Remark 4.23.** A unique solution of (2) that exhibits regularity is distinct from a solution of (2) which is the only regular solution. Theorem 4.22 gives conditions for the former (i.e., there cannot exist another solution, regular or not).

**Example 4.24.** Consider (3) and let $N_1 := (-1, 1) \times (-1, 1) \times (-1, 1) \times (0, 2)$. For any $(p, \eta_x, \eta_y) \in \pi_{p,x,y}N_1 = (-1, 1) \times (-1, 1) \times (0, 2)$,

$$f(\cdot, p, \eta_x, \eta_y) : t \mapsto -\chi_{(-1,0.5)}(t) + \chi_{(0.5,1)}(t)$$

is measurable on $\pi_t(N_1; (p, \eta_x, \eta_y)) = (-1, 1)$, where $\chi$ denotes the usual indicator function. For any $t \in \pi_t N_1 = (-1, 1)$,

$$f(t, \cdot, \cdot, \cdot) : (p, \eta_x, \eta_y) \mapsto \begin{cases} -1, & \text{if } t \in (-1, 0.5), \\ 0, & \text{if } t = 0.5, \\ 1, & \text{if } t \in (0.5, 1), \end{cases}$$

is continuous on $\pi_{p,x,y}(N_1; t) = (-1, 1) \times (-1, 1) \times (0, 2)$. For any $(t, p, \eta_x, \eta_y) \in N_1$,

$$|f(t, p, \eta_x, \eta_y)| \leq |\operatorname{sign}(t - 0.5)| + |(1.5|1 - \eta_y|^{\frac{1}{3}} - 1)H(t - 1)| \leq 1.$$

For any $(t, p_1, \eta_{x_1}, \eta_{y_1}), (t, p_2, \eta_{x_2}, \eta_{y_2}) \in N_1$,

$$|f(t, p_1, \eta_{x_1}, \eta_{y_1}) - f(t, p_2, \eta_{x_2}, \eta_{y_2})| = 0.$$

$f_0$ and $g$ are $PC^1$ (and therefore locally Lipschitz continuous) on $\pi_p N_1 = (-1, 1)$ and $N_1$, respectively.

Let $(t_0, p_0, x_0, y_0) := (0.5, -0.5, -0.5, 0.5) \in G_{\mathrm{C},0} \cap G_{\mathrm{R}} \cap N_1$. There exists a unique regular solution of (3) on $[0.5 - \alpha, 0.5 + \alpha] \times \{-0.5\}$ through $\{(0.5, -0.5, -0.5, 0.5)\}$ for some $\alpha > 0$ by Theorems 4.20 and 4.22; the mapping $\mathbf{z}^*$ as defined in Example 4.6 is such a solution (with $\alpha = 0.25$).

**Example 4.25.** Consider (3) and let $N_2 := \mathbb{R} \times (-1, 1) \times (-1, 1) \times (-2, 2)$. For any $(p, \eta_x, \eta_y) \in \pi_{p,x,y}N_2 = (-1, 1) \times (-1, 1) \times (-2, 2)$,

$$f(\cdot, p, \eta_x, \eta_y) : t \mapsto -\chi_{(-\infty,0.5)}(t) + \chi_{(0.5,1)}(t) + 1.5|1 - \eta_y|^{\frac{1}{3}}\chi_{[1,+\infty)}(t)$$

is measurable on $\pi_t(N_2; (p, \eta_x, \eta_y)) = \mathbb{R}$. For any $t \in \pi_t N_2 = \mathbb{R}$,

$$f(t,\cdot,\cdot,\cdot) : (p, \eta_x, \eta_y) \mapsto \begin{cases} -1, & \text{if } t < 0.5, \\ 0, & \text{if } t = 0.5, \\ 1, & \text{if } t \in (0.5, 1), \\ 1.5|1-\eta_y|^{\frac{1}{3}}, & \text{if } t \geq 1, \end{cases}$$

is continuous on $\pi_{p,x,y}(N_2; t) = (-1, 1) \times (-1, 1) \times (-2, 2)$. For any $(t, p, \eta_x, \eta_y) \in N_2$,

$$\begin{aligned} |f(t, p, \eta_x, \eta_y)| &\leq |\operatorname{sign}(t - 0.5)| + |(1.5|1-\eta_y|^{\frac{1}{3}} - 1)H(t-1)|, \\ &\leq 1 + 1.5|1-\eta_y|^{\frac{1}{3}} + 1, \\ &\leq 4.5. \end{aligned}$$

$f_0$ and $g$ are $PC^1$ (and therefore locally Lipschitz continuous) on $\pi_p N_2 = (-1, 1)$ and $N_2$, respectively.

Let $(t_0, p_0, x_0, y_0) := (1, 0, 0, 1) \in G_{\mathrm{C},0} \cap G_{\mathrm{R}} \cap N_2$. Then Theorem 4.20 is satisfied but Assumption 4.21 does not hold. Consequently, existence of a solution is guaranteed but its uniqueness need not hold as Theorem 4.22 is not applicable. Indeed, for each $\lambda \in [0, 1]$, $\mathbf{z}^{(\lambda)}$ as defined in Example 4.11 is a regular solution of (3) on $[0.5, 1.5] \times \{0\}$ through $\{(1, 0, 0, 1)\}$ in $N_2$ (i.e., $\alpha = 0.5$).

**Remark 4.26.** Non-uniqueness of $\mathbf{z}^{\dagger}$, as outlined in Example 4.8, on $[0, 1] \times \{-0.5\}$ through $\{(0.5, -0.5, -0.5, 0.5)\}$ is a consequence of loss of regularity at $t = 0.5$. Non-uniqueness of $\mathbf{z}^{(1)}$, as outlined in Example 4.11, on $[-0.5, 2] \times \{0\}$ through $\{(0, 0, 0, 1)\}$ is a consequence of the right-hand side function $f$.

*4.3. Continuation of Solutions*

Extended existence of solutions of (2) is detailed here. Analogous ODE system results can be found for the classical case (see Chapter 1, Section 3 in [59] and Chapter 1, Section 4 in [54]) and the Carathéodory ODE system case (see Chapter 2, Section 1 in [54] and Chapter 1, Section 1 in [53]).

**Definition 4.27.** Given a (regular) solution $\mathbf{z}$ of (2) on $T \times \{\mathbf{p}_0\}$ through $\{(t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0)\}$ in $N$, a mapping $\widetilde{\mathbf{z}}$ is called a *(regular) continuation of* $\mathbf{z}$ *on* $J$ *in* $N$ if $\tilde{\mathbf{z}}$ is a (regular) solution of (2) on $J \times \{\mathbf{p}_0\}$ through $\{(t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0)\}$ in $N$, $J \subset \pi_t N$ is a strict superset of $T$, and $\mathbf{z}(t, \mathbf{p}_0) = \widetilde{\mathbf{z}}(t, \mathbf{p}_0)$ for all $t \in T$.

By virtue of regularity, solutions can be extended.

**Theorem 4.28.** Let Assumptions 4.2 and 4.17 hold and $t_l, t_u \in \pi_t N$ such that $t_l < t_u$. Suppose that there exists a regular solution $\mathbf{z}$ of (2) on $[t_l, t_u] \times \{\mathbf{p}_0\}$ through $\{(t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0)\}$ in $N$ for some $(t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0) \in N$. Then there exist $\alpha > 0$ and a regular continuation $\widetilde{\mathbf{z}}$ of $\mathbf{z}$ on $[t_l - \alpha, t_u + \alpha]$ in $N$.

*Proof.* Define the following sets:

$$\begin{aligned} \Lambda &:= \{(t, \mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0)) : t \in [t_l, t_u]\} \subset \pi_{t,p,x} N, \\ \Omega &:= \{(t, \mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0), \mathbf{y}(t, \mathbf{p}_0)) : t \in [t_l, t_u]\} \subset N. \end{aligned}$$

The set $\Omega$ is compact since $t \mapsto (t, \mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0), \mathbf{y}(t, \mathbf{p}_0))$ is a continuous mapping on $[t_l, t_u]$. Note that $\pi_4 \partial \mathbf{g}(t, \mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y)$ is of maximal rank for all $(t, \mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y) \in \Omega$ by regularity and $\mathbf{g}(\Omega) = \{\mathbf{0}_{n_y}\}$ by consistency. Each point in $\Lambda$ is the projection of a unique point in $\Omega$ by Lemma 3.8.

Theorem 3.6 implies the existence of $\delta, \rho > 0$ and a mapping

$$\mathbf{r} : B_\delta(\Lambda) \subset \pi_{t,p,x} N \to \mathbb{R}^{n_y}$$

that is Lipschitz continuous on $B_\delta(\Lambda)$ such that, for all $(t, \mathbf{p}, \boldsymbol{\eta}_x) \in B_\delta(\Lambda)$, $(t, \mathbf{p}, \boldsymbol{\eta}_x, \mathbf{r}(t, \mathbf{p}, \boldsymbol{\eta}_x))$ is the unique vector in $B_\rho(\Omega) \subset N$ satisfying $\mathbf{0}_{n_y} = \mathbf{g}(t, \mathbf{p}, \boldsymbol{\eta}_x, \mathbf{r}(t, \mathbf{p}, \boldsymbol{\eta}_x))$ and $\pi_4 \partial \mathbf{g}(t, \mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y)$ is of maximal rank for all $(t, \mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y) \in B_\rho(\Omega)$. Then $\mathbf{y}(t, \mathbf{p}_0) = \mathbf{r}(t, \mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0))$ for all $t \in [t_l, t_u]$.

Define the following mappings:

$$\begin{aligned} \mathbf{q} &: B_\delta(\Lambda) \to \pi_{p,x,y} N : (t, \mathbf{p}, \boldsymbol{\eta}_x) \mapsto (\mathbf{p}_0, \boldsymbol{\eta}_x, \mathbf{r}(t, \mathbf{p}, \boldsymbol{\eta}_x)), \\ \bar{\mathbf{f}} &: B_\delta(\Lambda) \to \mathbb{R}^{n_p + n_x} : (t, \mathbf{p}, \boldsymbol{\eta}_x) \mapsto \begin{bmatrix} \mathbf{0}_{n_p} \\ \mathbf{f}(t, \mathbf{q}(t, \mathbf{p}, \boldsymbol{\eta}_x)) \end{bmatrix}, \end{aligned}$$

and note that

$$\{(t, \mathbf{q}(t, \mathbf{p}, \boldsymbol{\eta}_x)) : (t, \mathbf{p}, \boldsymbol{\eta}_x) \in B_\delta(\Lambda)\} \subset B_\rho(\Omega) \subset N. \tag{12}$$

Let $\mathbf{c}_0 := (\mathbf{p}_0, \mathbf{x}_0)$ and consider the following ODE system:

$$\begin{aligned} \dot{\mathbf{u}}(t) &= \bar{\mathbf{f}}(t, \mathbf{u}(t)), \\ \mathbf{u}(t_0) &= \mathbf{c}_0. \end{aligned} \tag{13}$$

It follows from the Carathéodory existence conditions of $\mathbf{f}$ on $N$ and (12) that, for each $(t, \mathbf{p}, \boldsymbol{\eta}_x) \in B_\delta(\Lambda)$,

$$\|\bar{\mathbf{f}}(t, \mathbf{p}, \boldsymbol{\eta}_x)\| = \|\mathbf{f}(t, \mathbf{p}, \boldsymbol{\eta}_x, \mathbf{r}(t, \mathbf{p}, \boldsymbol{\eta}_x))\| \leq m_{\mathbf{f}}(t).$$

For each $t \in \pi_t B_\delta(\Lambda) \setminus Z_{\mathbf{f}} = (t_l - \delta, t_u + \delta) \setminus Z_{\mathbf{f}}$, the mapping

$$\bar{\mathbf{f}}(t, \cdot, \cdot) \equiv \begin{bmatrix} \mathbf{0}_{n_p} \\ \mathbf{f}(t, \cdot, \cdot, \mathbf{r}(t, \cdot, \cdot)) \end{bmatrix} : \pi_{p,x}(B_\delta(\Lambda); t) \subset \pi_{p,x} N \to \mathbb{R}^{n_p + n_x}$$

is continuous on its domain by continuity of $\mathbf{r}$ on $B_\delta(\Lambda)$, (12), and continuity of the mapping $\mathbf{f}(t, \cdot, \cdot, \cdot)$ on $\pi_{p,x,y}(N; t) \supset \pi_{p,x,y}(B_\rho(\Omega); t)$ for each $t \in \pi_t N \setminus Z_{\mathbf{f}} \supset (t_l - \delta, t_u + \delta) \setminus Z_{\mathbf{f}}$. For each $(\mathbf{p}, \boldsymbol{\eta}_x) \in \pi_{p,x} B_\delta(\Lambda)$, the mapping

$$\bar{\mathbf{f}}(\cdot, \mathbf{p}, \boldsymbol{\eta}_x) \equiv \begin{bmatrix} \mathbf{0}_{n_p} \\ \mathbf{f}(\cdot, \mathbf{p}, \boldsymbol{\eta}_x, \mathbf{r}(\cdot, \mathbf{p}, \boldsymbol{\eta}_x)) \end{bmatrix} : \pi_t(B_\delta(\Lambda); (\mathbf{p}, \boldsymbol{\eta}_x)) \subset \pi_t N \to \mathbb{R}^{n_p + n_x}$$

is measurable on $\pi_t(B_\delta(\Lambda); (\mathbf{p}, \boldsymbol{\eta}_x))$ by Lemma 4.19 since the set $\pi_t(B_\delta(\Lambda); (\mathbf{p}, \boldsymbol{\eta}_x))$ is open (and therefore measurable) by Lemma 2.2, the mapping $t \mapsto (\mathbf{p}, \boldsymbol{\eta}_x, \mathbf{r}(t, \mathbf{p}, \boldsymbol{\eta}_x))$ is continuous (and thus measurable) on $\pi_t(B_\delta(\Lambda); (\mathbf{p}, \boldsymbol{\eta}_x))$, and $\mathbf{f}$ satisfies the Carathéodory existence conditions on

$$N \supset B_\rho(\Omega) \supset \{(t, \mathbf{p}, \boldsymbol{\eta}_x, \mathbf{r}(t, \mathbf{p}, \boldsymbol{\eta}_x)) : t \in \pi_t(B_\delta(\Lambda); (\mathbf{p}, \boldsymbol{\eta}_x))\}$$

by assumption.

Therefore, $\bar{\mathbf{f}}$ satisfies the Carathéodory existence conditions on $B_\delta(\Lambda)$, which is open by construction and connected since it is path connected. Equation (13) admits the solution

$$\mathbf{u} : [t_l, t_u] \to \mathbb{R}^{n_p \times n_x} : t \mapsto \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{x}(t, \mathbf{p}_0) \end{bmatrix},$$

such that $\{(t, \mathbf{u}(t)) : t \in [t_l, t_u]\} \subset B_\delta(\Lambda) \subset \pi_{t,p,x} N$. By virtue of Theorem 1.3 in Chapter 2, Section 1 [54], there exists $\alpha > 0$ and absolutely continuous mapping $\widetilde{\mathbf{u}} : [t_l - \alpha, t_u + \alpha] \to \mathbb{R}^{n_p \times n_x}$ which is a continuation of $\mathbf{u}$ on $[t_l - \alpha, t_u + \alpha]$ in $N$:

$$\widetilde{\mathbf{u}}(t) = \mathbf{c}_0 + \int_{t_0}^{t} \bar{\mathbf{f}}(s, \widetilde{\mathbf{u}}(s)) \mathrm{d}s, \quad \forall t \in [t_l - \alpha, t_u + \alpha],$$

and $\{(t, \widetilde{\mathbf{u}}(t)) : t \in [t_l - \alpha, t_u + \alpha]\} \subset B_\delta(\Lambda) \subset \pi_{t,p,x} N$. Define the mappings

$$\begin{aligned} (\widetilde{\boldsymbol{\rho}}, \widetilde{\mathbf{x}}) &: [t_l - \alpha, t_u + \alpha] \times \{\mathbf{p}_0\} \to \mathbb{R}^{n_p + n_x} : (t, \mathbf{p}) \mapsto \widetilde{\mathbf{u}}(t), \\ \widetilde{\mathbf{y}} &: [t_l - \alpha, t_u + \alpha] \times \{\mathbf{p}_0\} \to \mathbb{R}^{n_y} : (t, \mathbf{p}) \mapsto \mathbf{r}(t, \widetilde{\boldsymbol{\rho}}(t, \mathbf{p}), \widetilde{\mathbf{x}}(t, \mathbf{p})). \end{aligned}$$

By inspection, $\widetilde{\boldsymbol{\rho}}(t, \mathbf{p}) = \mathbf{p}_0$ for all $(t, \mathbf{p}) \in [t_l - \alpha, t_u + \alpha] \times \{\mathbf{p}_0\}$ and the mapping $\widetilde{\mathbf{x}}(\cdot, \mathbf{p}_0)$ is absolutely continuous on $[t_l - \alpha, t_u + \alpha]$ and satisfies

$$\widetilde{\mathbf{x}}(t, \mathbf{p}_0) = \mathbf{f}_0(\mathbf{p}_0) + \int_{t_0}^{t} \mathbf{f}(s, \mathbf{p}_0, \widetilde{\mathbf{x}}(s, \mathbf{p}_0), \mathbf{r}(s, \mathbf{p}_0, \widetilde{\mathbf{x}}(s, \mathbf{p}_0))) \mathrm{d}s, \quad \forall t \in [t_l - \alpha, t_u + \alpha].$$

From

$$\{(t, \mathbf{p}_0, \widetilde{\mathbf{x}}(t, \mathbf{p}_0)) : t \in [t_l - \alpha, t_u + \alpha]\} = \{(t, \widetilde{\mathbf{u}}(t)) : t \in [t_l - \alpha, t_u + \alpha]\} \subset B_\delta(\Lambda) \subset \pi_{t,p,x} N,$$

it follows that

$$\begin{aligned}\{(t,\mathbf{p}_0,\widetilde{\mathbf{x}}(t,\mathbf{p}_0),\widetilde{\mathbf{y}}(t,\mathbf{p}_0)) : t\in[t_l-\alpha,t_u+\alpha]\} &= \{(t,\mathbf{q}(t,\mathbf{p}_0,\widetilde{\mathbf{x}}(t,\mathbf{p}_0))) : t\in[t_l-\alpha,t_u+\alpha]\},\\ &\subset B_\rho(\Omega),\\ &\subset N.\end{aligned}$$

Since $\pi_4\partial\mathbf{g}(t,\mathbf{p},\boldsymbol{\eta}_x,\boldsymbol{\eta}_y)$ is of maximal rank for all $(t,\mathbf{p},\boldsymbol{\eta}_x,\boldsymbol{\eta}_y)\in B_\rho(\Omega)$,

$$\{(t,\mathbf{p}_0,\widetilde{\mathbf{x}}(t,\mathbf{p}_0),\widetilde{\mathbf{y}}(t,\mathbf{p}_0)) : t\in[t_l-\alpha,t_u+\alpha]\}\subset G_{\mathrm{R}}.$$

Lemma 2.5 implies that $\widetilde{\mathbf{y}}(\cdot,\mathbf{p}_0)$ is an absolutely continuous mapping on $[t_l-\alpha,t_u+\alpha]$. Furthermore,

$$\mathbf{0}_{n_y}=\mathbf{g}(t,\mathbf{p}_0,\widetilde{\mathbf{x}}(t,\mathbf{p}_0),\mathbf{r}(t,\mathbf{p}_0,\widetilde{\mathbf{x}}(t,\mathbf{p}_0)))=\mathbf{g}(t,\mathbf{p}_0,\widetilde{\mathbf{x}}(t,\mathbf{p}_0),\widetilde{\mathbf{y}}(t,\mathbf{p}_0)),\quad \forall t\in[t_l-\alpha,t_u+\alpha].$$

Then, by maximal rankness of $\pi_4\partial\mathbf{g}(t,\mathbf{p},\boldsymbol{\eta}_x,\boldsymbol{\eta}_y)$ for all $(t,\mathbf{p},\boldsymbol{\eta}_x,\boldsymbol{\eta}_y)\in B_\rho(\Omega)$, $(\widetilde{\mathbf{x}},\widetilde{\mathbf{y}})$ is a regular solution of (2) on $[t_l-\alpha,t_u+\alpha]\times\{\mathbf{p}_0\}$ through $\{(t_0,\mathbf{p}_0,\mathbf{x}_0,\mathbf{y}_0)\}$ in $N$. By continuation, $\mathbf{u}(t,\mathbf{p}_0)=\widetilde{\mathbf{u}}(t,\mathbf{p}_0)$ for all $t\in[t_l,t_u]$. It follows that $\mathbf{x}(t,\mathbf{p}_0)=\widetilde{\mathbf{x}}(t,\mathbf{p}_0)$ for all $t\in[t_l,t_u]$. Moreover,

$$\mathbf{y}(t,\mathbf{p}_0)=\mathbf{r}(t,\mathbf{p}_0,\mathbf{x}(t,\mathbf{p}_0))=\mathbf{r}(t,\mathbf{p}_0,\widetilde{\mathbf{x}}(t,\mathbf{p}_0))=\widetilde{\mathbf{y}}(t,\mathbf{p}_0),\quad\forall t\in[t_l,t_u].$$

Therefore, since $[t_l-\alpha,t_u+\alpha]\subset\pi_tN$ is a strict superset of $[t_l,t_u]$, $\widetilde{\mathbf{z}}\equiv(\widetilde{\mathbf{x}},\widetilde{\mathbf{y}})$ is a regular continuation of $\mathbf{z}$ on $[t_l-\alpha,t_u+\alpha]\times\{\mathbf{p}_0\}$ in $N$. □

Regular solutions can be continued to a maximal regular continuation, proved using Zorn's Lemma along the lines of Section 1.7 in [60].

**Definition 4.29.** Let $\mathbf{z}$ be a (regular) solution of (2) on $T\times\{\mathbf{p}_0\}$ through $\{(t_0,\mathbf{p}_0,\mathbf{x}_0,\mathbf{y}_0)\}$ in $N$ and $\mathbf{z}_{\max}$ be a (regular) continuation of $\mathbf{z}$ on $T_{\max}$ in $N$ which has no (regular) continuation for any superset of $T_{\max}$ contained in $\pi_tN$. The mapping $\mathbf{z}_{\max}$ is called a *maximal (regular) continuation of* $\mathbf{z}$ and $T_{\max}$ is called a *maximal horizon of (regular) existence of* $\mathbf{z}$ *in* $N$.

**Theorem 4.30.** Let Assumptions 4.2 and 4.17 hold and $t_l,t_u\in\pi_tN$ such that $t_l<t_u$. Suppose that there exists a regular solution $\mathbf{z}$ of (2) on $[t_l,t_u]\times\{\mathbf{p}_0\}$ through $\{(t_0,\mathbf{p}_0,\mathbf{x}_0,\mathbf{y}_0)\}$ in $N$ for some $(t_0,\mathbf{p}_0,\mathbf{x}_0,\mathbf{y}_0)\in N$. Then there exist $t_L\in\mathbb{R}\cup\{-\infty\}$ and $t_U\in\mathbb{R}\cup\{+\infty\}$ satisfying $t_L<t_l<t_u<t_U$ and a maximal regular continuation $\mathbf{z}_{\max}$ of $\mathbf{z}$ on $(t_L,t_U)$ in $N$.

*Proof.* Define the set of augmented graphs of regular continuations of $\mathbf{z}$ as follows:

$$\Gamma_{\mathrm{ext}}:=\{\{(t,\mathbf{p}_0,\widetilde{\mathbf{z}}(t,\mathbf{p}_0)) : t\in\widetilde{T}\} : \widetilde{\mathbf{z}}\text{ is a regular continuation of }\mathbf{z}\text{ on }\widetilde{T}\text{ in }N\},\tag{14}$$

which is nonempty by Theorem 4.28. Define the generalized inequality $\preceq$ as follows: for any $\Phi,\Psi\in\Gamma_{\mathrm{ext}}$, $\Phi\preceq\Psi$ if and only if $\Phi\subset\Psi$. Let $\Omega_1,\Omega_2,\Omega_3\in\Gamma_{\mathrm{ext}}$. Reflexivity and antisymmetry are trivially satisfied. Suppose that $\Omega_1\preceq\Omega_2$ and $\Omega_2\preceq\Omega_3$. Then $\Omega_1\subset\Omega_2\subset\Omega_3$, implying transitivity is satisfied since $\Omega_1\preceq\Omega_3$. Hence, $\preceq$ is a partial order on $\Gamma_{\mathrm{ext}}$.

Let $\Gamma^*_{\mathrm{ext}}:=\{\Omega_{(i)} : i\in A\}$ be any nonempty totally ordered subset of $\Gamma_{\mathrm{ext}}$, where $A$ is an index set (possibly uncountable). For each $i\in A$, let $T_{(i)}:=\pi_t\Omega_{(i)}$. For each $i\in A$ and each $(t,\mathbf{p})\in T_{(i)}\times\{\mathbf{p}_0\}$, $\pi_{x,y}(\Omega_{(i)};t)$ is a singleton by construction of $\Gamma_{\mathrm{ext}}$. For each $i\in A$, let

$$\mathbf{z}_{(i)} : T_{(i)}\times\{\mathbf{p}_0\}\to\pi_{x,y}N,$$

denote the function which maps $(t,\mathbf{p})\in T_{(i)}\times\{\mathbf{p}_0\}$ to the single element of $\pi_{x,y}(\Omega_{(i)};t)$. Then $\mathbf{z}_{(i)}$ is a regular continuation of $\mathbf{z}$ on $T_{(i)}\supset[t_l,t_u]$ in $N$ by definition of $\Gamma_{\mathrm{ext}}$. Since $\Gamma^*_{\mathrm{ext}}$ is a totally ordered set, comparability implies that either

$$\Omega_{(j)}=\{(t,\mathbf{p}_0,\mathbf{z}_{(j)}(t,\mathbf{p}_0)) : t\in T_{(j)}\}\subset\{(t,\mathbf{p}_0,\mathbf{z}_{(k)}(t,\mathbf{p}_0)) : t\in T_{(k)}\}=\Omega_{(k)}$$

or

$$\Omega_{(k)}=\{(t,\mathbf{p}_0,\mathbf{z}_{(k)}(t,\mathbf{p}_0)) : t\in T_{(k)}\}\subset\{(t,\mathbf{p}_0,\mathbf{z}_{(j)}(t,\mathbf{p}_0)) : t\in T_{(j)}\}=\Omega_{(j)}$$

holds for each $k, j \in A$. Then for any $j \in A$,

$$\{(t, \mathbf{p}_0, \mathbf{z}_{(j)}(t, \mathbf{p}_0)) : t \in T_{(j)}\} \subset \bigcup_{i \in A} \{(t, \mathbf{p}_0, \mathbf{z}_{(i)}(t, \mathbf{p}_0)) : t \in T_{(i)}\} = \bigcup_{i \in A} \Omega_{(i)} =: \Omega_u \in \Gamma_{\text{ext}}.$$

By construction, $\Omega_{(i)} \subset \Omega_u$ for all $i \in A$, from which it follows that $\Omega_u$ is an upper bound on $\Gamma^*_{\text{ext}}$. By Zorn's Lemma (Lemma 2.12), $\Gamma_{\text{ext}}$ contains maximal elements. Let $\Omega_{\max}$ be one such maximal element. Let $T_{\max} := \pi_t \Omega_{\max}$ and

$$\mathbf{z}_{\max} : T_{\max} \times \{\mathbf{p}_0\} \to \pi_{x,y} N$$

denote the function which maps $(t, \mathbf{p}) \in T_{\max} \times \{\mathbf{p}_0\}$ to the single element of $\pi_{x,y}(\Omega_{\max}; t)$. Then, by construction of $\Gamma_{\text{ext}}$, $\mathbf{z}_{\max}$ is a maximal regular continuation of $\mathbf{z}$ on $T_{\max}$ in $N$.

It is claimed that $T_{\max}$ is open. If not then suppose, without loss of generality, that it includes its right endpoint: $T_{\max} = (t_*, t^*]$ for some $t_* < t^*$ satisfying $t_*, t^* \in \pi_t N$. Then $(t^*, \mathbf{p}_0, \mathbf{z}_{\max}(t^*, \mathbf{p}_0)) \in G_{\text{C}} \cap G_{\text{R}} \cap N$ and Theorem 4.20 may be applied to assert the existence of $\alpha > 0$ and a regular solution $\mathbf{z}^*$ of (2) on $[t^* - \alpha, t^* + \alpha] \times \{\mathbf{p}_0\}$ through $\{(t^*, \mathbf{p}_0, \mathbf{x}_{\max}(t^*, \mathbf{p}_0), \mathbf{y}_{\max}(t^*, \mathbf{p}_0))\}$ in $N$. Concatenate $\mathbf{z}_{\max}$ and $\mathbf{z}^*$ to construct a new solution; let

$$\mathbf{z}^\dagger : (t_*, t^* + \alpha] \times \{\mathbf{p}_0\} \to \pi_{x,y} N : (t, \mathbf{p}) \mapsto \begin{cases} \mathbf{z}_{\max}(t, \mathbf{p}), & \text{if } t \in (t_*, t^*], \\ \mathbf{z}^*(t, \mathbf{p}), & \text{if } t \in (t^*, t^* + \alpha]. \end{cases}$$

It follows that $\mathbf{z}^\dagger$ is a regular continuation of $\mathbf{z}_{\max}$ on $(t_*, t^* + \alpha]$ in $N$, contradicting the fact that $\mathbf{z}_{\max}$ is a maximal regular continuation of $\mathbf{z}$ in $N$. Therefore, it must be that $T_{\max} =: (t_L, t_U)$ is open.

□

**Remark 4.31.** Elements of the set of augmented graphs $\Gamma_{\text{ext}}$ defined in (14) need not be comparable under the partial ordering outlined in the proof of Theorem 4.30. However, augmented graphs of any nonempty totally ordered subset of $\Gamma_{\text{ext}}$ are comparable by definition.

**Remark 4.32.** Since uniqueness of $\mathbf{z}$ is not demanded in Theorem 4.28, any continuation depends on the original solution of interest and need not be unique itself. This is also true in Theorem 4.30; both the maximal regular continuation and maximal horizon of existence depend on the original solution and any subsequent continuations. This can be seen in the proof of Theorem 4.30 since $\mathbf{z}_{\max}$ and $(t_L, t_U)$ both depend on the choice of $\Omega_{\max}$.

**Example 4.33.** Consider (3) and the regular solution $\mathbf{z}^*$ of (3) on $[0.25, 0.75] \times \{-0.5\}$ through $\{(0.5, -0.5, -0.5, 0.5)\}$ as outlined in Example 4.6. Observe that

$$[0.25, 0.75] \times \{-0.5\} \times \mathbf{z}^*([0.25, 0.75] \times \{-0.5\}) \subset N_2 := \mathbb{R} \times (-1, 1) \times (-1, 1) \times (-2, 2),$$

and note that Assumptions 4.2 and 4.17 hold on $N_2$ (see Example 4.25). Theorem 4.28 is applicable; the mappings

$$\widetilde{\mathbf{z}}^{(0)} : [-0.5, 1.5] \times \{-0.5\} \to \mathbb{R}^2 : (t, p) \mapsto \begin{cases} (-t, 1 + t), & \text{if } t \in [-0.5, 0], \\ \mathbf{z}^{(0)}(t, 0), & \text{if } t \in (0, 1.5], \end{cases}$$

$$\widetilde{\mathbf{z}}^{(1)} : [-0.5, 1.5] \times \{-0.5\} \to \mathbb{R}^2 : (t, p) \mapsto \begin{cases} (-t, 1 + t), & \text{if } t \in [-0.5, 0], \\ \mathbf{z}^{(1)}(t, 0), & \text{if } t \in (0, 1.5], \end{cases}$$

where $\mathbf{z}^{(0)}$ and $\mathbf{z}^{(1)}$ are outlined in Example 4.11, are regular continuations of $\mathbf{z}^*$ on $[-0.5, 1.5]$ in $N_2$ (i.e., $\alpha = 0.75$).

Moreover, in accordance with Theorem 4.30, the mappings

$$\widetilde{\mathbf{z}}^{(0)}_{\max} : (-1, 2) \times \{-0.5\} \to \mathbb{R}^2 : (t, p) \mapsto \begin{cases} (-t, 1 + t), & \text{if } t \in (-1, 0], \\ \mathbf{z}^{(0)}(t, 0), & \text{if } t \in (0, 2), \end{cases}$$

$$\widetilde{\mathbf{z}}^{(1)}_{\max} : (-1, +\infty) \times \{-0.5\} \to \mathbb{R}^2 : (t, p) \mapsto \begin{cases} (-t, 1 + t), & \text{if } t \in (-1, 0], \\ \mathbf{z}^{(1)}(t, 0), & \text{if } t \in (0, 2], \\ (0, 1), & \text{if } t > 2, \end{cases}$$

are maximal regular continuations of $\mathbf{z}^*$ in $N_2$ with associated maximal horizons of regular existence given by $T^{(0)}_{\max} := (-1, 2)$ and $T^{(1)}_{\max} := (-1, +\infty)$, respectively. $\widetilde{\mathbf{z}}^{(0)}_{\max}$ has no regular continuation for any superset of $T^{(0)}_{\max}$ by loss of regularity;

$$\lim_{t \to -1^+} (t, -0.5, \widetilde{\mathbf{z}}^{(0)}_{\max}(t, -0.5)) = (-1, -0.5, 1, 0) \notin \mathcal{G}_{\mathrm{R}}$$

and

$$\lim_{t \to 2^-} (t, -0.5, \widetilde{\mathbf{z}}^{(0)}_{\max}(t, -0.5)) = (2, -0.5, 1, 0) \notin \mathcal{G}_{\mathrm{R}}.$$

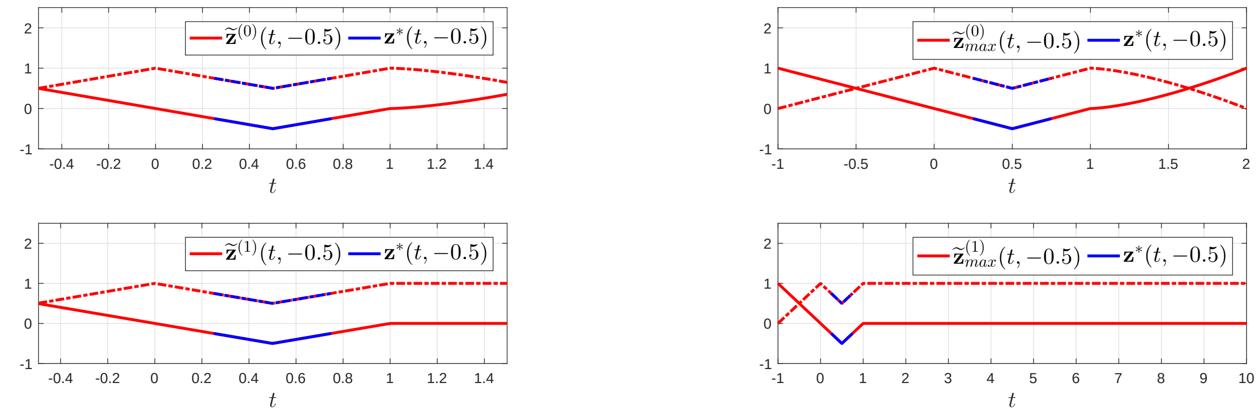
Similarly,

$$\lim_{t \to -1^+} (t, -0.5, \widetilde{\mathbf{z}}^{(1)}_{\max}(t, -0.5)) = (-1, -0.5, 1, 0) \notin \mathcal{G}_{\mathrm{R}},$$

while

$$\lim_{t \to +\infty} (t, -0.5, \widetilde{\mathbf{z}}^{(1)}_{\max}(t, -0.5)) \notin N.$$

The continuations of $\mathbf{z}^*$ in this example are illustrated in Figure 5.




(a) $\mathbf{z}^*(t, -0.5)$, $\widetilde{\mathbf{z}}^{(0)}(t, -0.5)$, and $\widetilde{\mathbf{z}}^{(1)}(t, -0.5)$ vs. $t$.

(b) $\mathbf{z}^*(t, -0.5)$, $\widetilde{\mathbf{z}}^{(0)}_{\max}(t, -0.5)$, and $\widetilde{\mathbf{z}}^{(1)}_{\max}(t, -0.5)$ vs. $t$.

Figure 5: Graphs of Example 4.33; solid and dotted lines correspond to the differential and algebraic variables associated with a solution, respectively.

### *4.4. Parametric Dependence of Solutions*

Continuous and differentiable dependence of solutions of classical ODEs on initial values and parameters is summarized in [54, 59]. For a review of pertinent results regarding continuous dependence of solutions of Carathéodory ODEs, see Chapter 2, Section 4 in [54], or, more recently, Chapter 1, Section 1 in [53]. Theorem 4.13 establishes the consistent initialization of (2) in a neighborhood of a parameter value, given regularity and consistency of a point. To address existence of solutions and their behavior with respect to parameters in said neighborhood, the following continuous dependence result is provided.

**Theorem 4.34.** Let Assumptions 4.1, 4.2, and 4.17 hold. Let $t_f \in \mathbb{R}$ such that $t_f > t_0$ and $[t_0, t_f] \subset \pi_t N$. Suppose that there exists a unique regular solution of (2) on $[t_0, t_f] \times \{\mathbf{p}_0\}$ through $\{(t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0)\}$ in $N$ for some $(t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0) \in N$. Then there exist a neighborhood $N(\mathbf{p}_0) \subset \pi_p N$ of $\mathbf{p}_0$, $\Omega_0 \subset G_{\mathrm{C},0} \cap G_{\mathrm{R}} \cap N$ containing $(t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0)$, and a regular solution $\mathbf{z}$ of (2) on $[t_0, t_f] \times N(\mathbf{p}_0)$ through $\Omega_0$ in $N$. Moreover, for any $\epsilon > 0$ there exists $\alpha > 0$ satisfying $B_\alpha(\mathbf{p}_0) \subset N(\mathbf{p}_0)$ such that for any $t \in [t_0, t_f]$,

$$\|\mathbf{z}(t, \mathbf{p}) - \mathbf{z}(t, \mathbf{p}_0)\| < \epsilon, \quad \forall \mathbf{p} \in B_\alpha(\mathbf{p}_0),$$

and therefore, for each $t \in [t_0, t_f]$, the mapping $\mathbf{z}_t \equiv \mathbf{z}(t, \cdot)$ is continuous at $\mathbf{p}_0$.

*Proof.* Let $\widetilde{\mathbf{z}} \equiv (\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})$ denote the unique regular solution of (2) on $[t_0, t_f] \times \{\mathbf{p}_0\}$ through $\{(t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0)\}$ in $N$. Following the proof of Theorem 4.28 with $t_l$ and $t_u$ replaced by $t_0$ and $t_f$, respectively, define the following mappings:

$$\mathbf{q} : B_\delta(\Lambda) \to \pi_{p,x,y}N : (t, \mathbf{p}, \boldsymbol{\eta}_x) \mapsto (\mathbf{p}, \boldsymbol{\eta}_x, \mathbf{r}(t, \mathbf{p}, \boldsymbol{\eta}_x)),$$
$$\bar{\mathbf{f}} : B_\delta(\Lambda) \to \mathbb{R}^{n_p+n_x} : (t, \mathbf{p}, \boldsymbol{\eta}_x) \mapsto \begin{bmatrix} \mathbf{0}_{n_p} \\ \mathbf{f}(t, \mathbf{q}(t, \mathbf{p}, \boldsymbol{\eta}_x)) \end{bmatrix}.$$

It can be shown that $\bar{\mathbf{f}}$ satisfies the Carathéodory existence conditions on the open and connected set $B_\delta(\Lambda)$. The absolutely continuous mapping

$$t \mapsto \begin{bmatrix} \mathbf{p}_0 \\ \widetilde{\mathbf{x}}(t, \mathbf{p}_0) \end{bmatrix}$$

is the unique solution of the ODE system

$$\begin{aligned} \dot{\mathbf{u}}(t, \mathbf{c}) &= \bar{\mathbf{f}}(t, \mathbf{u}(t, \mathbf{c})), \quad \text{a.e. } t \in [t_0, t_f], \\ \mathbf{u}(t_0, \mathbf{c}) &= \mathbf{c}, \end{aligned} \tag{15}$$

at $\mathbf{c} := (\mathbf{p}_0, \mathbf{x}_0)$. By Theorem 4.2 in Chapter 2, Section 4 [54], there exist $\xi > 0$ and a mapping $\mathbf{u} : [t_0, t_f] \times B_\xi(\mathbf{p}_0, \mathbf{x}_0) \to \mathbb{R}^{n_p+n_x}$ such that $B_\xi(\mathbf{p}_0, \mathbf{x}_0) \subset \pi_{p,x}(B_\delta(\Lambda); t_0) \subset \pi_{p,x}N$ since $(\mathbf{p}_0, \mathbf{x}_0) \in \pi_{p,x}N$ and for each $\mathbf{c} \in B_\xi(\mathbf{p}_0, \mathbf{x}_0)$, $\mathbf{u}(\cdot, \mathbf{c})$ is an absolutely continuous mapping which satisfies

$$\mathbf{u}(t, \mathbf{c}) = \mathbf{c} + \int_{t_0}^{t} \bar{\mathbf{f}}(s, \mathbf{u}(s, \mathbf{c}))\mathrm{d}s, \quad \forall t \in [t_0, t_f],$$

and

$$\{(t, \mathbf{u}(t, \mathbf{c})) : (t, \mathbf{c}) \in [t_0, t_f] \times B_\xi(\mathbf{p}_0, \mathbf{x}_0)\} \subset B_\delta(\Lambda) \subset \pi_{t,p,x}N.$$

By local Lipschitz continuity of $\mathbf{f}_0$, there exists $k_{\mathbf{f}_0} \geq 0$ such that

$$\|\mathbf{f}_0(\mathbf{p}) - \mathbf{x}_0\| \leq k_{\mathbf{f}_0}\|\mathbf{p} - \mathbf{p}_0\|, \quad \forall \mathbf{p} \in \bar{B}_{0.5\xi}(\mathbf{p}_0) \subset \pi_p N \subset D_p. \tag{16}$$

Letting $\beta := 0.5\xi/(k_{\mathbf{f}_0} + 1)$, it follows that $B_\beta(\mathbf{p}_0) \subset \bar{B}_{0.75\xi}(\mathbf{p}_0)$ and

$$\begin{aligned} \|(\mathbf{p}, \mathbf{f}_0(\mathbf{p})) - (\mathbf{p}_0, \mathbf{x}_0)\| &\leq \|\mathbf{p} - \mathbf{p}_0\| + \|\mathbf{f}_0(\mathbf{p}) - \mathbf{x}_0\|, \\ &\leq \frac{\xi}{2(k_{\mathbf{f}_0} + 1)} + k_{\mathbf{f}_0}\|\mathbf{p} - \mathbf{p}_0\|, \\ &< \xi, \quad \forall \mathbf{p} \in B_\beta(\mathbf{p}_0). \end{aligned}$$

Therefore, $\{(\mathbf{p}, \mathbf{f}_0(\mathbf{p})) : \mathbf{p} \in B_\beta(\mathbf{p}_0)\} \subset B_\xi(\mathbf{p}_0, \mathbf{x}_0)$. Let

$$(\boldsymbol{\rho}, \mathbf{x}) : [t_0, t_f] \times B_\beta(\mathbf{p}_0) \to \mathbb{R}^{n_p+n_x} : (t, \mathbf{p}) \mapsto \mathbf{u}(t, (\mathbf{p}, \mathbf{f}_0(\mathbf{p}))).$$

By inspection, $\boldsymbol{\rho}(t, \mathbf{p}) = \mathbf{p}$ for all $(t, \mathbf{p}) \in [t_0, t_f] \times B_\beta(\mathbf{p}_0)$ and for each $\mathbf{p} \in B_\beta(\mathbf{p}_0)$, the mapping $\mathbf{x}(\cdot, \mathbf{p})$ is absolutely continuous and satisfies

$$\mathbf{x}(t, \mathbf{p}) = \mathbf{f}_0(\mathbf{p}) + \int_{t_0}^{t} \mathbf{f}(s, \mathbf{p}, \mathbf{x}(s, \mathbf{p}), \mathbf{r}(s, \mathbf{p}, \mathbf{x}(s, \mathbf{p})))\mathrm{d}s, \quad \forall t \in [t_0, t_f].$$

Define the mapping

$$\mathbf{y} : [t_0, t_f] \times B_\beta(\mathbf{p}_0) \to \mathbb{R}^{n_y} : (t, \mathbf{p}) \mapsto \mathbf{r}(t, \mathbf{p}, \mathbf{x}(t, \mathbf{p})).$$

Since $\mathbf{y}(t, \mathbf{p}) = \mathbf{r}(t, \boldsymbol{\rho}(t, \mathbf{p}), \mathbf{x}(t, \mathbf{p}))$ for all $(t, \mathbf{p}) \in [t_0, t_f] \times B_\beta(\mathbf{p}_0)$, Lemma 2.5 implies that $\mathbf{y}(\cdot, \mathbf{p})$ is an absolutely continuous mapping on $[t_0, t_f]$ for each $\mathbf{p} \in B_\beta(\mathbf{p}_0)$. Furthermore,

$$\begin{aligned} \mathbf{0}_{n_y} &= \mathbf{g}(t, \mathbf{p}, \mathbf{x}(t, \mathbf{p}), \mathbf{r}(t, \mathbf{p}, \mathbf{x}(t, \mathbf{p}))), \\ &= \mathbf{g}(t, \mathbf{p}, \mathbf{x}(t, \mathbf{p}), \mathbf{y}(t, \mathbf{p})), \quad \forall (t, \mathbf{p}) \in [t_0, t_f] \times B_\beta(\mathbf{p}_0), \end{aligned}$$

and recall $\pi_4 \partial \mathbf{g}(t, \mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y)$ is of maximal rank for all $(t, \mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y) \in B_\rho(\Omega)$. Hence,

$$\{(t, \mathbf{p}, \mathbf{x}(t, \mathbf{p}), \mathbf{y}(t, \mathbf{p})) : (t, \mathbf{p}) \in [t_0, t_f] \times B_\beta(\mathbf{p}_0)\} \subset B_\rho(\Omega) \subset G_{\mathrm{R}} \cap N,$$

and $(\mathbf{x}, \mathbf{y})$ is a regular solution of (2) on $[t_0, t_f] \times B_\beta(\mathbf{p}_0)$ through

$$\Omega_0 := \{(t, \mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y) : t = t_0, \mathbf{p} \in B_\beta(\mathbf{p}_0), \boldsymbol{\eta}_x = \mathbf{f}_0(\mathbf{p}), \boldsymbol{\eta}_y = \mathbf{r}(t_0, \mathbf{p}, \mathbf{f}_0(\mathbf{p}))\} \subset G_{\mathrm{C},0} \cap G_{\mathrm{R}} \cap N$$

in $N$. The first conclusion of the theorem holds with $N(\mathbf{p}_0) := B_\beta(\mathbf{p}_0)$.

Choose any $\epsilon > 0$. By virtue of Theorem 4.2 in Chapter 2, Section 4 [54], there exists $\xi^* \in (0, \xi]$ such that for any $t \in [t_0, t_f]$,

$$\|\mathbf{u}(t, \mathbf{c}) - \mathbf{u}(t, \mathbf{c}_0)\| < \frac{\epsilon}{2(1 + k_{\mathbf{r}})}, \quad \forall \mathbf{c} \in B_{\xi^*}(\mathbf{p}_0, \mathbf{x}_0),$$

where $\mathbf{c}_0 := (\mathbf{p}_0, \mathbf{x}_0)$. Let $\alpha^* := 0.5\xi^*/(k_{\mathbf{f}_0} + 1)$. Then $B_{\alpha^*}(\mathbf{p}_0) \subset \bar{B}_{0.75\xi^*}(\mathbf{p}_0) \subset \bar{B}_{0.75\xi}(\mathbf{p}_0)$ and

$$\|(\mathbf{p}, \mathbf{f}_0(\mathbf{p})) - (\mathbf{p}_0, \mathbf{x}_0)\| \le \frac{\xi^*}{2(k_{\mathbf{f}_0} + 1)} + k_{\mathbf{f}_0}\|\mathbf{p} - \mathbf{p}_0\| < \xi^*, \quad \forall \mathbf{p} \in B_{\alpha^*}(\mathbf{p}_0).$$

Hence, $\{(\mathbf{p}, \mathbf{f}_0(\mathbf{p})) : \mathbf{p} \in B_{\alpha^*}(\mathbf{p}_0)\} \subset B_{\xi^*}(\mathbf{p}_0, \mathbf{x}_0)$ and it follows that, for any $t \in [t_0, t_f]$,

$$\begin{aligned}
\|\mathbf{x}(t, \mathbf{p}) - \mathbf{x}(t, \mathbf{p}_0)\| &\le \|(\mathbf{p}, \mathbf{x}(t, \mathbf{p})) - (\mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0))\|, \\
&= \|\mathbf{u}(t, (\mathbf{p}, \mathbf{f}_0(\mathbf{p}))) - \mathbf{u}(t, (\mathbf{p}_0, \mathbf{x}_0))\|, \\
&< \frac{\epsilon}{2(1 + k_{\mathbf{r}})}.
\end{aligned}$$

Then for any $t \in [t_0, t_f]$,

$$\|\mathbf{p} - \mathbf{p}_0\| < \alpha := \min\left\{\alpha^*, \frac{\epsilon}{2(1 + k_{\mathbf{r}})}\right\}$$

implies that

$$\begin{aligned}
\|\mathbf{z}(t, \mathbf{p}) - \mathbf{z}(t, \mathbf{p}_0)\| &\le \|\mathbf{x}(t, \mathbf{p}) - \mathbf{x}(t, \mathbf{p}_0)\| + \|\mathbf{y}(t, \mathbf{p}) - \mathbf{y}(t, \mathbf{p}_0)\|, \\
&= \|\mathbf{x}(t, \mathbf{p}) - \mathbf{x}(t, \mathbf{p}_0)\| + \|\mathbf{r}(t, \mathbf{p}, \mathbf{x}(t, \mathbf{p})) - \mathbf{r}(t, \mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0))\|, \\
&\le \|\mathbf{x}(t, \mathbf{p}) - \mathbf{x}(t, \mathbf{p}_0)\| + k_{\mathbf{r}}(\|\mathbf{p} - \mathbf{p}_0\| + \|\mathbf{x}(t, \mathbf{p}) - \mathbf{x}(t, \mathbf{p}_0)\|), \\
&= (1 + k_{\mathbf{r}})\|\mathbf{x}(t, \mathbf{p}) - \mathbf{x}(t, \mathbf{p}_0)\| + k_{\mathbf{r}}\|\mathbf{p} - \mathbf{p}_0\|, \\
&< \epsilon.
\end{aligned}$$

□

Behavior of DAE solutions along a fixed parameter value is characterized as follows.

**Proposition 4.35.** Let the conditions of Theorem 4.34 hold and $\mathbf{z}$ be a corresponding regular solution of (2) on $[t_0, t_f] \times N(\mathbf{p}_0)$ through $\Omega_0$ in $N$. Then for any $\epsilon > 0$ there exists $\alpha > 0$ such that for any $t^* \in [t_0, t_f]$ and any $\mathbf{p} \in N(\mathbf{p}_0)$,

$$\|\mathbf{z}(t, \mathbf{p}) - \mathbf{z}(t^*, \mathbf{p})\| < \epsilon, \quad \forall t \in B_\alpha(t^*) \cap [t_0, t_f].$$

*Proof.* This result is shown along the lines of the proof of Lemma 2 in Chapter 1, Section 1 [53]: define the mapping

$$\varphi : [t_0, t_f] \to \mathbb{R}_+ : t \mapsto \int_{t_0}^{t} m_{\mathbf{f}}(s)\mathrm{d}s,$$

which is absolutely continuous (and therefore uniformly continuous) on $[t_0, t_f]$ since $m_{\mathbf{f}}$ is Lebesgue integrable on $[t_0, t_f]$. By the proof of Theorem 4.34, $\{(t, \mathbf{p}, \mathbf{x}(t, \mathbf{p})) : (t, \mathbf{p}) \in [t_0, t_f] \times N(\mathbf{p}_0)\} \subset B_\delta(\Lambda)$ and $\mathbf{y}(t, \mathbf{p}) = \mathbf{r}(t, \mathbf{p}, \mathbf{x}(t, \mathbf{p}))$

for all $(t, \mathbf{p}) \in [t_0, t_f] \times N(\mathbf{p}_0)$, where $\mathbf{r}$ is a Lipschitz continuous function on $B_\delta(\Lambda)$, with Lipschitz constant $k_\mathbf{r} \geq 0$. For any $\mathbf{p} \in N(\mathbf{p}_0)$ and any closed interval $[t_1, t_2] \subset [t_0, t_f]$,

$$\begin{aligned}
&\|\mathbf{z}(t_2, \mathbf{p}) - \mathbf{z}(t_1, \mathbf{p})\| \\
&\leq \|\mathbf{x}(t_2, \mathbf{p}) - \mathbf{x}(t_1, \mathbf{p})\| + \|\mathbf{y}(t_2, \mathbf{p}) - \mathbf{y}(t_1, \mathbf{p})\|, \\
&= \left\| \int_{t_1}^{t_2} \mathbf{f}(s, \mathbf{p}, \mathbf{x}(s, \mathbf{p}), \mathbf{r}(s, \mathbf{p}, \mathbf{x}(s, \mathbf{p}))) \mathrm{d}s \right\| + \|\mathbf{r}(t_2, \mathbf{p}, \mathbf{x}(t_2, \mathbf{p})) - \mathbf{r}(t_1, \mathbf{p}, \mathbf{x}(t_1, \mathbf{p}))\|, \\
&\leq \int_{t_1}^{t_2} \|\mathbf{f}(s, \mathbf{p}, \mathbf{x}(s, \mathbf{p}), \mathbf{r}(s, \mathbf{p}, \mathbf{x}(s, \mathbf{p})))\| \mathrm{d}s + k_\mathbf{r} \|(t_2, \mathbf{p}, \mathbf{x}(t_2, \mathbf{p})) - (t_1, \mathbf{p}, \mathbf{x}(t_1, \mathbf{p}))\|, \\
&\leq \int_{t_1}^{t_2} \|\mathbf{f}(s, \mathbf{p}, \mathbf{x}(s, \mathbf{p}), \mathbf{r}(s, \mathbf{p}, \mathbf{x}(s, \mathbf{p})))\| \mathrm{d}s + k_\mathbf{r} \left( t_2 - t_1 + \int_{t_1}^{t_2} \|\mathbf{f}(s, \mathbf{p}, \mathbf{x}(s, \mathbf{p}), \mathbf{r}(s, \mathbf{p}, \mathbf{x}(s, \mathbf{p})))\| \mathrm{d}s \right), \\
&\leq (1 + k_\mathbf{r}) \int_{t_1}^{t_2} m_\mathbf{f}(s) \mathrm{d}s + k_\mathbf{r}(t_2 - t_1), \\
&= (1 + k_\mathbf{r})(\varphi(t_2) - \varphi(t_1)) + k_\mathbf{r}(t_2 - t_1).
\end{aligned}$$

By uniform continuity of $\varphi$, for any $\epsilon > 0$ there exists $\alpha^* > 0$ such that for any $t^* \in [t_0, t_f]$,

$$|\varphi(t) - \varphi(t^*)| < \frac{\epsilon}{2(1 + k_\mathbf{r})}, \quad \forall t \in B_{\alpha^*}(t^*) \cap [t_0, t_f].$$

Hence, for any $t^* \in [t_0, t_f]$ and any $\mathbf{p} \in N(\mathbf{p}_0)$,

$$\|\mathbf{z}(t, \mathbf{p}) - \mathbf{z}(t^*, \mathbf{p})\| \leq (1 + k_\mathbf{r})|\varphi(t) - \varphi(t^*)| + k_\mathbf{r}|t - t^*| < \epsilon, \quad \forall t \in B_\alpha(t^*) \cap [t_0, t_f],$$

where $\alpha := \min\{\alpha^*, 0.5\epsilon/(1 + k_\mathbf{r})\}$. □

Solutions inherit uniqueness and Lipschitzian dependence in parameters under an additional Carathéodory-style uniqueness assumption.

**Theorem 4.36.** Let Assumptions 4.1, 4.2, 4.17, and 4.21 hold. Let $t_f \in \mathbb{R}$ such that $t_f > t_0$ and $[t_0, t_f] \subset \pi_t N$. Suppose that there exists a regular solution of (2) on $[t_0, t_f] \times \{\mathbf{p}_0\}$ through $\{(t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0)\}$ in $N$ for some $(t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0) \in N$. Then there exist a neighborhood $N(\mathbf{p}_0) \subset \pi_p N$ of $\mathbf{p}_0$, $\Omega_0 \subset G_{\mathrm{C},0} \cap G_\mathrm{R} \cap N$ containing $(t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0)$, and a unique regular solution $\mathbf{z}$ of (2) on $[t_0, t_f] \times N(\mathbf{p}_0)$ through $\Omega_0$ in $N$. Moreover, for each $t \in [t_0, t_f]$, the mapping $\mathbf{z}_t \equiv \mathbf{z}(t, \cdot)$ is Lipschitz continuous on a neighborhood of $\mathbf{p}_0$, with a Lipschitz constant that is independent of $t$.

*Proof.* Let $\widetilde{\mathbf{z}}$ denote the regular solution of (2) on $[t_0, t_f] \times \{\mathbf{p}_0\}$ through $\{(t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0)\}$ in $N$. By Theorem 4.22, $\widetilde{\mathbf{z}}$ is a unique regular solution of (2) on $[t_0, t_f] \times \{\mathbf{p}_0\}$ through $\{(t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0)\}$ in $N$. By Theorem 4.34, there exist a neighborhood $N(\mathbf{p}_0) \subset \pi_p N$ of $\mathbf{p}_0$, $\Omega_0 \subset G_{\mathrm{C},0} \cap G_\mathrm{R} \cap N$ containing $(t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0)$, and a regular solution $\mathbf{z}$ of (2) on $[t_0, t_f] \times N(\mathbf{p}_0)$ through $\Omega_0$ in $N$. Another application of Theorem 4.22 yields that $\mathbf{z}$ is a unique regular solution of (2) on $[t_0, t_f] \times N(\mathbf{p}_0)$ through $\Omega_0$ in $N$.

It is possible to show that $\mathbf{x}_t$ is Lipschitz continuous on $N(\mathbf{p}_0)$ similarly as in the proof of Theorem 4.1 in [19]: for any $t \in [t_0, t_f]$ and any $\mathbf{p}_1, \mathbf{p}_2 \in N(\mathbf{p}_0)$,

$$\begin{aligned}
\|\mathbf{x}(t, \mathbf{p}_1) - \mathbf{x}(t, \mathbf{p}_2)\| &= \left\| \mathbf{f}_0(\mathbf{p}_1) + \int_{t_0}^{t} \mathbf{f}(s, \mathbf{x}(s, \mathbf{p}_1), \mathbf{r}(s, \mathbf{x}(s, \mathbf{p}_1))) \mathrm{d}s - \mathbf{f}_0(\mathbf{p}_2) - \int_{t_0}^{t} \mathbf{f}(s, \mathbf{x}(s, \mathbf{p}_2), \mathbf{r}(s, \mathbf{y}(s, \mathbf{p}_2))) \mathrm{d}s \right\|, \\
&\leq \|\mathbf{f}_0(\mathbf{p}_1) - \mathbf{f}_0(\mathbf{p}_2)\| + \int_{t_0}^{t} k_\mathbf{f}(s)(1 + k_\mathbf{r}) \|\mathbf{x}(s, \mathbf{p}_1) - \mathbf{x}(s, \mathbf{p}_2)\| \mathrm{d}s, \\
&\leq \|\mathbf{f}_0(\mathbf{p}_1) - \mathbf{f}_0(\mathbf{p}_2)\| \exp\left( \int_{t_0}^{t} k_\mathbf{f}(s)(1 + k_\mathbf{r}) \mathrm{d}s \right), \\
&\leq k_{\mathbf{f}_0} k_\mathbf{x} \|\mathbf{p}_1 - \mathbf{p}_2\|,
\end{aligned}$$

where $k_{\mathbf{x}} := \exp((1+k_{\mathbf{r}})\int_{t_0}^{t_f} k_{\mathbf{f}}(s)\mathrm{d}s)$ by the version of Gronwall's Inequality outlined in [61], and $k_{\mathbf{f}_0} \geq 0$ is a Lipschitz constant for $\mathbf{f}_0$ on $N(\mathbf{p}_0)$ as outlined in (16). Then, for any $t \in [t_0, t_f]$,

$$\begin{aligned}\|\mathbf{y}(t,\mathbf{p}_1) - \mathbf{y}(t,\mathbf{p}_2)\| &= \|\mathbf{r}(t,\mathbf{p}_1,\mathbf{x}(t,\mathbf{p}_1)) - \mathbf{r}(t,\mathbf{p}_2,\mathbf{x}(t,\mathbf{p}_2))\|,\\ &\leq k_{\mathbf{r}}\|(\mathbf{p}_1,\mathbf{x}(t,\mathbf{p}_1)) - (\mathbf{p}_2,\mathbf{x}(t,\mathbf{p}_2))\|,\\ &\leq k_{\mathbf{r}}(1+k_{\mathbf{f}_0}k_{\mathbf{x}})\|\mathbf{p}_1-\mathbf{p}_2\|, \quad \forall \mathbf{p}_1,\mathbf{p}_2 \in N(\mathbf{p}_0).\end{aligned}$$

Hence, for any $t \in [t_0, t_f]$, $\|\mathbf{z}(t,\mathbf{p}_1) - \mathbf{z}(t,\mathbf{p}_2)\| \leq \max\{k_{\mathbf{f}_0}k_{\mathbf{x}}, k_{\mathbf{r}}(1+k_{\mathbf{f}_0}k_{\mathbf{x}})\}\|\mathbf{p}_1-\mathbf{p}_2\|$ for any $\mathbf{p}_1,\mathbf{p}_2 \in N(\mathbf{p}_0)$. □

**Example 4.37.** Consider the following semi-explicit DAEs with explicit parametric dependence:

$$\begin{aligned}\dot{x}(t,p) &= 0.5\mathrm{sign}(1-t)\max\{0,p\}y(t,p),\\ 0 &= |x(t,p)| + |y(t,p)| - 1,\\ x(0,p) &= \arctan(p).\end{aligned} \tag{17}$$

The right-hand side functions are given by

$$\begin{aligned}f &: \mathbb{R}^4 \to \mathbb{R} : (t,p,\eta_x,\eta_y) \mapsto 0.5\mathrm{sign}(1-t)\max\{0,p\}\eta_y,\\ g &: \mathbb{R}^4 \to \mathbb{R} : (t,p,\eta_x,\eta_y) \mapsto |\eta_x| + |\eta_y| - 1,\\ f_0 &: \mathbb{R} \to \mathbb{R} : p \mapsto \arctan(p),\end{aligned}$$

$f_0$ is $C^1$ on $\mathbb{R}$, $g$ is $PC^1$ on $\mathbb{R}^4$, and $f$ satisfies Assumptions 4.17 and 4.21 on $N := \mathbb{R}^4$.

Let $[t_0, t_f] := [0, 2]$ and $(p_0, x_0, y_0) := (0, 0, 1)$. The mapping

$$\widetilde{\mathbf{z}} \equiv (\widetilde{x}, \widetilde{y}) : [t_0, t_f] \times \{p_0\} \to \mathbb{R}^2 : (t,p) \mapsto (0,1)$$

is a unique solution of (17) on $[t_0, t_f] \times \{p_0\}$ through $\{(t_0, p_0, x_0, y_0)\}$ in $N$. This solution is regular as

$$\pi_4 \partial g(t,p,\widetilde{x}(t,p),\widetilde{y}(t,p)) = \{1\}, \quad \forall (t,p) \in [t_0,t_f] \times \{p_0\}$$

since $\widetilde{y}(t,p) > 0$ for all $(t,p) \in [t_0,t_f] \times \{p_0\}$. By Theorem 4.36, there exist a neighborhood $N(p_0) \subset \pi_p N$ of $p_0$, $\Omega_0 \subset \mathcal{G}_{\mathrm{C},0} \cap G_{\mathrm{R}} \cap N$, and a unique regular solution of (17) on $[t_0, t_f] \times N(p_0)$ through $\Omega_0$ in $N$. Indeed, let $N(p_0) := (-0.5, 0.5)$ and $\beta : (0, 0.5) \to (0, 1) : p \mapsto (\arctan(p) - 1)\exp(-0.5p) + 1$, then

$$\mathbf{z} \equiv (x,y) : [t_0,t_f] \times N(p_0) \to \mathbb{R}^2 : (t,p) \mapsto \begin{cases} \begin{bmatrix} (\arctan(p)-1)\exp(-0.5pt)+1 \\ (1-\arctan(p))\exp(-0.5pt) \end{bmatrix}, & \text{if } (t,p) \in [0,1) \times (0,0.5),\\ \begin{bmatrix} (\beta(p)-1)\exp(0.5p(t-1))+1 \\ (1-\beta(p))\exp(0.5p(t-1)) \end{bmatrix}, & \text{if } (t,p) \in [1,2] \times (0,0.5),\\ \begin{bmatrix} \arctan(p) \\ 1+\arctan(p) \end{bmatrix}, & \text{if } (t,p) \in [0,2] \times (-0.5,0], \end{cases}$$

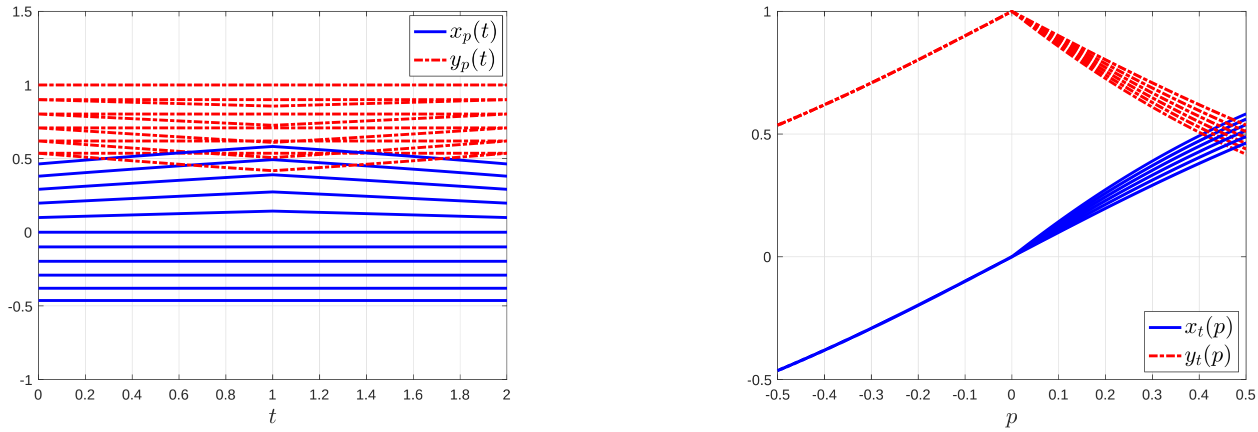
is a unique regular solution of (17) on $[t_0, t_f] \times N(p_0)$ through

$$\Omega_0 := \{(t,p,\eta_x,\eta_y) : t = 0, p \in (-0.5, 0.5), \eta_x = \arctan(p), \eta_y = 1 - |\arctan(p)|\}$$

in $N$. Moreover, for each $t \in [t_0, t_f]$, the mapping $\mathbf{z}_t \equiv \mathbf{z}(t,\cdot)$ is indeed Lipschitz continuous on a neighborhood of $p_0$ (with a Lipschitz constant independent of $t$) since $\mathbf{z}$ is $PC^1$ on $[t_0, t_f] \times N(p_0)$ and therefore locally Lipschitz continuous on $[t_0, t_f] \times N(p_0)$. See Figure 6 for an illustration.

## 5. Discussion

Existence, uniqueness, and parametric dependence of solutions of nonsmooth DAEs have been analyzed using generalized derivatives. For these efforts, an extended implicit function theorem has been developed for locally Lipschitz continuous algebraic equations. A Carathéodory ODE system, which is equivalent to the Carathéodory




(a) $\mathbf{z}(t,p)$ vs. $t$ for various values of $-0.5 < p < 0.5$.

(b) $\mathbf{z}(t,p)$ vs. $p$ for various values of $0 \leq t \leq 2$.

Figure 6: Graphs of Example 4.37.

DAE system of interest on an open and connected set, is formulated using this extended implicit function theorem. The right-hand side function of said equivalent Carathéodory ODE system is shown to inherit the Carathéodory-style conditions. In doing so, the theoretical groundwork has been laid for future theoretical (e.g., sensitivity analysis [48]) and numerical investigations. Consistent initialization and local existence of solutions have been ascertained in terms of consistency and regularity of the DAE system at initialization. Given a regular solution on a finite horizon and at one parameter value, a regular solution is immediately furnished on a neighborhood of said parameter value and the solution's parametric dependence has been shown to be continuous or Lipschitzian. Directions for future work include extensions of the above work to DAEs with discontinuities with respect to the independent variable appearing in the algebraic equations and "high-index" nonsmooth DAE systems.

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