Forward Parametric Sensitivity Functions for Carathéodory Index-1 Semi-Explicit Differential-Algebraic Equations

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Abstract

Nonsmooth equation-solving and optimization algorithms which require local sensitivity information are extended to systems with nonsmooth parametric differential-algebraic equations (DAEs) embedded. Nonsmooth DAEs refers here to semi-explicit DAEs with algebraic equations satisfying local Lipschitz continuity and differential right-hand side functions satisfying Carathéodory-like conditions. Using lexicographic differentiation, an auxiliary nonsmooth DAE system is obtained whose unique solution furnishes the desired parametric sensitivities. More specifically, lexicographic derivatives of solutions of nonsmooth parametric DAEs are obtained. Lexicographic derivatives have been shown to be elements of the plenary hull of the (Clarke) generalized Jacobian and thus computationally relevant in the aforementioned algorithms. To accomplish this goal, the lexicographic smoothness of an extended implicit function is proved. Moreover, these generalized derivative elements can be calculated in tractable ways thanks to recent advancements in nonsmooth analysis. Forward sensitivity functions for nonsmooth parametric DAEs are therefore characterized, extending the classical sensitivity results for smooth parametric DAEs.

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1. Introduction

Algorithms for nonsmooth equation-solving (e.g., semismooth Newton methods [1, 2] and LP-Newton methods [3]) and nonsmooth optimization (e.g., bundle methods for local optimization [4–6]) require sensitivity information for which many current theoretical and computational approaches are lacking. Recent progress has been made in tractable algorithms [7] for obtaining elements of a class of generalized derivative, using lexicographic differentiation [8] to calculate lexicographic directional derivatives, as introduced in [7]. Applicable to lexicographically smooth functions (which includes all differentiable functions, convex functions, and PC^1 functions in the sense of Scholtes [9]), this approach has been used to furnish computationally relevant generalized derivatives for parametric ordinary-differential equations (ODEs) with nonsmooth right-hand sides [10]; hybrid systems, inverse functions, and implicit functions [11]; ODEs with linear programs embedded [12]; and nonsmooth optimal control problems with nonsmooth ODEs embedded [13].

With applications in mechanical, electrical, and chemical engineering, DAEs (also called singular or descriptor systems) have become a widely applied modeling tool [14]. Narrowing the focus more, nonsmooth DAEs provide a natural modeling framework for a number of physical phenomena found in engineering and applied mathematics such as campaign continuous pharmaceutical manufacturing (see, e.g., [15–17]). In this paper, generalized derivative notions from nonsmooth analysis are used (for background, the reader is referred to [9, 18–20] and the references therein). Elements of the plenary hull of Clarke's generalized Jacobian comprise the desired sensitivity information for the nonsmooth algorithms described earlier. As DAEs pose a number of theoretical and numerical difficulties over ODEs (see, e.g., [14, 21–24] and the

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references therein), the extension of the aforementioned lexicographic differentiation theory to nonsmooth DAEs requires careful consideration.

Numerous studies have been completed on forward and adjoint sensitivities of smooth DAEs (see, e.g., [25, 26] and the references therein), hybrid and discontinuous systems (see, e.g., [27–29]), and oscillating systems [30, 31]. However, the theoretical tools and findings in these works are not applicable here due to incompatible assumptions. Clarke first derived a result on generalized Jacobians of solutions of nonsmooth parametric ODEs (Theorem 7.4.1 in [18]). Pang and Stewart (Theorem 11 and Corollary 12 in [32]) proved that such generalized Jacobian supersets are linear Newton approximations (LNAs, see [19] for details) when the ODE right-hand side functions are semismooth in the sense of Qi [2]. Pang and Stewart [32] then applied their ODE sensitivity results to differential variational inequalities (DVIs), as introduced in [33], with differentiable ODE right-hand side functions and differentiable variational condition functions; the authors calculated directional derivatives of local solutions of DVIs and obtained LNAs of the solution map about an initial data point. As DVIs can be expressed as a class of DAEs with specialized structure, the results in [32] are restricted to a subclass of nonsmooth DAEs with differentiable ODE right-hand side functions and nonsmooth algebraic equations. Furthermore, LNAs have been shown to not necessarily satisfy desirable properties which are satisfied by generalized Jacobians, such as LNAs of differentiable functions containing elements other than the Jacobian evaluated at said point (see Example 4.2 in [10] and Example 1.1 in [12]) and LNAs of convex scalar-valued functions including elements that are not subgradients [10].

Khan and Barton [10] derived a method for obtaining lexicographic derivatives of the unique solution of parametric Carathéodory ODEs from the unique solution of an auxiliary ODE system obtained via the lexicographic directional derivative chain rule [7]. This work is a natural extension of the classical sensitivity results for smooth parametric ODE systems obtained via the classical chain rule (see, e.g., Chapter V in [34]). As a subset of the plenary Jacobian, elements of the lexicographic subdifferential have been shown to be computationally relevant in many applications [10], including the nonsmooth algorithms detailed earlier. Moreover, as a key property of the lexicographic directional derivative is that it satisfies strict calculus rules, the implementation of a vector forward mode of automatic differentiation to calculate elements of the plenary Jacobian is therefore possible [7].

The main contribution of the current article is the development of a suitable theory for obtaining generalized derivative elements of solutions of nonsmooth parametric DAEs. In the spirit of [10], lexicographic derivatives (and therefore elements of the plenary Jacobian) of unique solutions of Carathéodory index-1 semi-explicit DAEs are obtained from the unique solution of an auxiliary nonsmooth DAE system via the lexicographic directional derivative chain rule. First, we derive the lexicographic smoothness of the extended implicit function constructed in [35] inherited from lexicographic smoothness of the participating functions. In doing so, it is possible to formulate the nonsmooth DAEs as equivalent parametric Carathéodory ODEs on an open and connected set containing the unique solution. The sensitivity theory developed here applies to DAEs for which existing methods fail and, thanks to the strict calculus rules of the lexicographic directional derivative, lays the theoretical groundwork upon which efficient numerical implementations can be designed. Methods for nonsmooth equation-solving and nonsmooth optimization are thus extended to systems with nonsmooth parametric DAEs embedded.

The rest of this article is organized as follows. Necessary background in nonsmooth analysis is presented in section 2. Lexicographic smoothness of extended implicit functions is proved in section 2.5. Generalized derivatives of nonsmooth DAEs are calculated in section 3; forward sensitivities are found for semi-explicit index-1 DAEs. Examples are given in section 4 and concluding remarks are provided in section 5.

2. Mathematical Background

Relevant preliminaries are discussed in this section.

2.1. Preliminaries

The notational conventions here echo those set out in [7, 10]. The set of positive integers is denoted by $\mathbb{N}$ and the set of nonnegative real numbers is denoted by $\mathbb{R}_+$. The vector space $\mathbb{R}^n$ is equipped with the Euclidean norm $\|\cdot\|$ and the vector space $\mathbb{R}^{m\times n}$ is equipped with the corresponding induced norm. Sets are denoted by uppercase letters (e.g., H), matrices in $\mathbb{R}^{m\times n}$ and matrix-valued functions are denoted by uppercase boldface letters (e.g., $\mathbf{H}$), elements of $\mathbb{R}$ and scalar-valued functions are denoted by lowercase letters (e.g., h), and vectors in $\mathbb{R}^n$ and vector-valued functions are denoted by lowercase boldface letters (e.g., $\mathbf{h}$). The zero vector in $\mathbb{R}^n$ is denoted by $\mathbf{0}_n$, the $m \times n$ zero matrix is denoted by $\mathbf{0}_{m\times n}$, and the $n \times n$ identity matrix is denoted by $\mathbf{I}_n$. A well-defined vertical block matrix (or vector):

$$\begin{bmatrix} \mathbf{H}_1 \\ \mathbf{H}_2 \end{bmatrix}$$

can be written as $(\mathbf{H}_1, \mathbf{H}_2)$. The i^{th} component of a vector $\mathbf{h}$ is denoted by h_i. Parenthetical subscripts may be used to indicate the column vector of a matrix (e.g., the matrix $\mathbf{H}$ has the k^{th} column $\mathbf{h}_{(k)}$), or to indicate a sequence of vectors or vector-valued functions. Parenthetical superscripts (e.g., $\mathbf{h}^{(k)}$) are used for lexicographic differentiation. A *neighborhood of* $\mathbf{h} \in \mathbb{R}^n$ is a set of points $B_\delta(\mathbf{h})$ (the open ball of radius δ centered at $\mathbf{h}$) for some $\delta > 0$. A *neighborhood of* $H \subset \mathbb{R}^n$ is given by $B_\delta(H) := \cup_{\mathbf{h}\in H} B_\delta(\mathbf{h})$ for some $\delta > 0$. The closed ball of radius $r > 0$ centered at $\mathbf{h}$ is denoted by $\bar{B}_r(\mathbf{h})$. Given a set $H \subset \mathbb{R}^n$, its convex hull is denoted by $\operatorname{conv} H$. A set of matrices $H \subset \mathbb{R}^{n\times n}$ is said to be of *maximal rank* if it contains no singular matrices.

Given $n_x, n_y, n_z \in \mathbb{N}$ and $W \subset \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_z}$, the *projections of* W *onto* $\mathbb{R}^{n_x}$ and $\mathbb{R}^{n_x} \times \mathbb{R}^{n_y}$ are given by, respectively,

$$\begin{aligned} \pi_x W &:= \{\boldsymbol{\eta}_x \in \mathbb{R}^{n_x} : \exists(\boldsymbol{\eta}_x, \boldsymbol{\eta}_y, \boldsymbol{\eta}_z) \in W\} \subset \mathbb{R}^{n_x}, \\ \pi_{x,y} W &:= \{(\boldsymbol{\eta}_x, \boldsymbol{\eta}_y) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} : \exists(\boldsymbol{\eta}_x, \boldsymbol{\eta}_y, \boldsymbol{\eta}_z) \in W\} \subset \mathbb{R}^{n_x} \times \mathbb{R}^{n_y}. \end{aligned}$$

The *shadows of* W *at* $\mathbf{y} \in \pi_y W$ *with respect to* $\mathbb{R}^{n_x}$ and $\mathbb{R}^{n_x} \times \mathbb{R}^{n_z}$ are given by, respectively,

$$\begin{aligned} \pi_x(W; \mathbf{y}) &:= \pi_x\{(\boldsymbol{\eta}_x, \boldsymbol{\eta}_y, \boldsymbol{\eta}_z) \in W : \boldsymbol{\eta}_y = \mathbf{y}\} \subset \mathbb{R}^{n_x}, \\ \pi_{x,z}(W; \mathbf{y}) &:= \pi_{x,z}\{(\boldsymbol{\eta}_x, \boldsymbol{\eta}_y, \boldsymbol{\eta}_z) \in W : \boldsymbol{\eta}_y = \mathbf{y}\} \subset \mathbb{R}^{n_x} \times \mathbb{R}^{n_z}. \end{aligned}$$

The *shadow of* W *at* $(\mathbf{x}, \mathbf{y}) \in \pi_{x,y} W$ *with respect to* $\mathbb{R}^{n_z}$ is given by

$$\pi_z(W; (\mathbf{x}, \mathbf{y})) := \pi_z\{(\boldsymbol{\eta}_x, \boldsymbol{\eta}_y, \boldsymbol{\eta}_z) \in W : (\boldsymbol{\eta}_x, \boldsymbol{\eta}_y) = (\mathbf{x}, \mathbf{y})\} \subset \mathbb{R}^{n_z}.$$

Given $n_q \in \mathbb{N}$, $W_x \subset \pi_x W$, $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in W$, and $\mathbf{f} : W \to \mathbb{R}^{n_q}$, the *cross-section of* $\mathbf{f}$ *at* $\mathbf{x} \in \pi_x W$ is given by

$$\mathbf{f}_{\mathbf{x}} : \pi_{y,z}(W; \mathbf{x}) \to \mathbb{R}^{n_q} : (\boldsymbol{\eta}_y, \boldsymbol{\eta}_z) \mapsto \mathbf{f}(\mathbf{x}, \boldsymbol{\eta}_y, \boldsymbol{\eta}_z).$$

The W_x*-blind cross-section of* $\mathbf{f}$ *at* $\mathbf{x}$ is given by

$$\mathbf{f}_{\mathbf{x}\setminus W_x} : \pi_{y,z}(W; \mathbf{x}) \to \mathbb{R}^{n_q} : (\boldsymbol{\eta}_y, \boldsymbol{\eta}_z) \mapsto \begin{cases} \mathbf{f}(\mathbf{x}, \boldsymbol{\eta}_y, \boldsymbol{\eta}_z), & \mathbf{x} \in \pi_x W \setminus W_x, \\ \mathbf{0}_{n_q}, & \mathbf{x} \in W_x. \end{cases}$$

The other non-vacuous projections, shadows, cross-sections and blind cross-sections are defined similarly.

Consider an open set $X \subset \mathbb{R}^n$ and a function $\mathbf{f} : X \to \mathbb{R}^m$. $\mathbf{f}$ is said to be (Fréchet) *differentiable* at $\mathbf{x} \in X$ if there exists a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ that satisfies

$$\mathbf{0}_m = \lim_{\boldsymbol{\alpha}\to\mathbf{0}_n} \frac{\mathbf{f}(\mathbf{x} + \boldsymbol{\alpha}) - (\mathbf{f}(\mathbf{x}) + \mathbf{A}\boldsymbol{\alpha})}{\|\boldsymbol{\alpha}\|}.$$

In this case, the matrix $\mathbf{A}$ is uniquely described by the above equation and is called the *Jacobian matrix*, denoted by $\mathbf{Jf}(\mathbf{x}) \in \mathbb{R}^{m\times n}$. $\mathbf{f}$ is said to be *differentiable on* X if $\mathbf{f}$ is differentiable at each point $\mathbf{x} \in X$. $\mathbf{f}$ is said to be *continuously differentiable* (C^1) *at* $\mathbf{x} \in X$ if $\mathbf{f}$ is differentiable on a neighborhood $N(\mathbf{x}) \subset X$ of $\mathbf{x}$ and $\mathbf{Jf} : N(\mathbf{x}) \to \mathbb{R}^{m\times n}$ is continuous on $N(\mathbf{x})$. $\mathbf{f}$ is said to be C^1 *on* X if $\mathbf{f}$ is C^1 at each point $\mathbf{x} \in X$. As defined by Scholtes [9], $\mathbf{f}$ is said to be *piecewise differentiable* (PC^1) at $\mathbf{x} \in X$ if there exist a neighborhood $N(\mathbf{x}) \subset X$ of $\mathbf{x}$ and a finite collection of functions of C^1 functions on $N(\mathbf{x})$, $\{\mathbf{f}_{(1)}, \ldots, \mathbf{f}_{(k)}\}$, such that $\mathbf{f}$ is continuous on $N(\mathbf{x})$ and

$$\mathbf{f}(\boldsymbol{\eta}) \in \{\mathbf{f}_{(i)}(\boldsymbol{\eta}) : i \in \{1, \ldots, k\}\}, \quad \forall \boldsymbol{\eta} \in N(\mathbf{x}).$$

$\mathbf{f}$ is said to be PC^1 *on* X if $\mathbf{f}$ is PC^1 at each point $\mathbf{x} \in X$.

2.2. Generalized Derivatives

Let $X \subset \mathbb{R}^n$ be open and $\mathbf{f} : X \to \mathbb{R}^m$ be locally Lipschitz continuous on X. It follows that $\mathbf{f}$ is differentiable at each point $\mathbf{x} \in X \setminus Z_{\mathbf{f}}$, where $Z_{\mathbf{f}} \subset X$ has zero (Lebesgue) measure, by Rademacher's Theorem. Clarke [18] established the following definitions and results concerning generalized derivatives. The *B-subdifferential* of $\mathbf{f}$ at $\mathbf{x} \in X$ is defined as

$$\partial_{\mathrm{B}}\mathbf{f}(\mathbf{x}) := \left\{ \lim_{i\to\infty} \mathbf{Jf}(\mathbf{x}_{(i)}) : \lim_{i\to\infty} \mathbf{x}_{(i)} = \mathbf{x}, \quad \mathbf{x}_{(i)} \in X \setminus Z_{\mathbf{f}}, \forall i \in \mathbb{N} \right\}.$$

The Clarke (generalized) *Jacobian* of $\mathbf{f}$ at $\mathbf{x} \in X$ is defined as

$$\partial\mathbf{f}(\mathbf{x}) := \operatorname{conv} \partial_{\mathrm{B}}\mathbf{f}(\mathbf{x}).$$

For a point $\mathbf{x} \in X$, $\partial_{\mathrm{B}}\mathbf{f}(\mathbf{x})$ is necessarily nonempty and compact, while $\partial\mathbf{f}(\mathbf{x})$ is necessarily nonempty, compact, and convex. If $\mathbf{f}$ is differentiable at $\mathbf{x} \in X$ then $\mathbf{Jf}(\mathbf{x}) \in \partial\mathbf{f}(\mathbf{x})$. If $\mathbf{f}$ is C^1 at $\mathbf{x}$ then $\partial\mathbf{f}(\mathbf{x}) = \partial_{\mathrm{B}}\mathbf{f}(\mathbf{x}) = \{\mathbf{Jf}(\mathbf{x})\}$.

Given $n_x, n_y, n_z, n_q \in \mathbb{N}$, $W \subset \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_z}$ open, and $\mathbf{g} : W \to \mathbb{R}^{n_q}$ Lipschitz continuous on a neighborhood of $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in W$, the Clarke (generalized) *Jacobian projections* of $\mathbf{g}$ at $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ are defined as

$$\begin{aligned}
\pi_1\partial\mathbf{g}(\mathbf{x}, \mathbf{y}, \mathbf{z}) &:= \left\{\mathbf{M} \in \mathbb{R}^{n_q \times n_x} : \exists [\mathbf{M} \quad \mathbf{N}_1 \quad \mathbf{N}_2] \in \partial\mathbf{g}(\mathbf{x}, \mathbf{y}, \mathbf{z})\right\}, \\
\pi_2\partial\mathbf{g}(\mathbf{x}, \mathbf{y}, \mathbf{z}) &:= \left\{\mathbf{M} \in \mathbb{R}^{n_q \times n_y} : \exists [\mathbf{N}_1 \quad \mathbf{M} \quad \mathbf{N}_2] \in \partial\mathbf{g}(\mathbf{x}, \mathbf{y}, \mathbf{z})\right\}, \\
\pi_{2,3}\partial\mathbf{g}(\mathbf{x}, \mathbf{y}, \mathbf{z}) &:= \left\{[\mathbf{M}_1 \quad \mathbf{M}_2] \in \mathbb{R}^{n_q \times (n_y + n_z)} : \exists [\mathbf{N} \quad \mathbf{M}_1 \quad \mathbf{M}_2] \in \partial\mathbf{g}(\mathbf{x}, \mathbf{y}, \mathbf{z})\right\},
\end{aligned}$$

with $\pi_3\partial\mathbf{g}(\mathbf{x}, \mathbf{y}, \mathbf{z})$, $\pi_{1,2}\partial\mathbf{g}(\mathbf{x}, \mathbf{y}, \mathbf{z})$, and $\pi_{1,3}\partial\mathbf{g}(\mathbf{x}, \mathbf{y}, \mathbf{z})$ defined similarly. If $\mathbf{g}$ is C^1 at $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ then

$$\pi_{2,3}\partial\mathbf{g}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \left\{ \left[\frac{\partial\mathbf{g}}{\partial\mathbf{y}}(\mathbf{x}, \mathbf{y}, \mathbf{z}) \quad \frac{\partial\mathbf{g}}{\partial\mathbf{z}}(\mathbf{x}, \mathbf{y}, \mathbf{z}) \right] \right\}.$$

The *plenary Jacobian* of $\mathbf{f}$ at $\mathbf{x} \in X$ [36] is defined as

$$\partial_{\mathrm{P}}\mathbf{f}(\mathbf{x}) := \{\mathbf{M} \in \mathbb{R}^{m\times n} : \forall \mathbf{d} \in \mathbb{R}^n, \exists \mathbf{H} \in \partial\mathbf{f}(\mathbf{x}) \text{ s.t. } \mathbf{Md} = \mathbf{Hd}\}.$$

As the name suggests, the plenary Jacobian of $\mathbf{f}$ at $\mathbf{x}$ is the plenary hull of its Clarke Jacobian at $\mathbf{x}$ (see [36] for details on plenary sets and plenary hulls); it is the intersection of all plenary supersets of $\partial\mathbf{f}(\mathbf{x})$, which includes all linear transformations for which images are indistinguishable. As demonstrated by Imbert [37], $\partial_{\mathrm{P}}\mathbf{f}(\mathbf{x})$ is nonempty, compact, convex, and satisfies

$$\partial\mathbf{f}(\mathbf{x}) \subset \partial_{\mathrm{P}}\mathbf{f}(\mathbf{x}) \subset \prod_{i=1}^{m} \partial f_i(\mathbf{x}).$$

Remark 2.1. As pointed out in [10], if $\min\{m, n\} = 1$ then $\partial\mathbf{f}(\mathbf{x}) = \partial_{\mathrm{P}}\mathbf{f}(\mathbf{x})$. Moreover, if $m = n$ and if $\partial\mathbf{f}(\mathbf{x})$ is of maximal rank then a similar relationship holds between images of inverses of elements of $\partial\mathbf{f}(\mathbf{x})$ and $\partial_{\mathrm{P}}\mathbf{f}(\mathbf{x})$:

$$\{\mathbf{H}^{-1}\mathbf{d} \in \mathbb{R}^n : \mathbf{H} \in \partial_{\mathrm{P}}\mathbf{f}(\mathbf{x})\} = \{\mathbf{H}^{-1}\mathbf{d} \in \mathbb{R}^n : \mathbf{H} \in \partial\mathbf{f}(\mathbf{x})\}, \quad \forall \mathbf{d} \in \mathbb{R}^n.$$

As a consequence of these observations, elements of the plenary Jacobian are no less useful than elements of the Clarke Jacobian in any of the following: bundle methods for finding local minima for nonsmooth nonlinear programs (since the objective function is scalar-valued), semismooth Newton methods, Clarke's mean value theorem (Proposition 2.6.5 in [18]), and Clarke's inverse function theorem (Theorem 7.1.1 in [18]).

2.3. Lexicographic Differentiation

Nesterov [8] introduced lexicographically smooth functions and the lexicographic (generalized) derivative. Given $X \subset \mathbb{R}^n$ open and $\mathbf{f} : X \to \mathbb{R}^m$, the *directional derivative* of $\mathbf{f}$ at $\mathbf{x} \in X$ in the direction $\mathbf{d} \in \mathbb{R}^n$ is given by

$$\mathbf{f}'(\mathbf{x}; \mathbf{d}) := \lim_{\alpha \downarrow 0} \frac{\mathbf{f}(\mathbf{x} + \alpha\mathbf{d}) - \mathbf{f}(\mathbf{x})}{\alpha},$$

if it exists. The function $\mathbf{f}$ is said to be *directionally differentiable at* $\mathbf{x}$ if $\mathbf{f}'(\mathbf{x};\mathbf{d})$ exists and is finite for all $\mathbf{d}\in\mathbb{R}^n$. Given that $\mathbf{f}$ is locally Lipschitz continuous on X, $\mathbf{f}$ is said to be *lexicographically smooth (L-smooth) at* $\mathbf{x}\in X$ if for any $k\in\mathbb{N}$ and any $\mathbf{M}:=[\mathbf{m}_{(1)}\ \cdots\ \mathbf{m}_{(k)}]\in\mathbb{R}^{n\times k}$, the following higher-order directional derivatives are well-defined:

$$\begin{aligned}
&\mathbf{f}^{(0)}_{\mathbf{x},\mathbf{M}}:\mathbb{R}^n\to\mathbb{R}^m:\mathbf{d}\mapsto\mathbf{f}'(\mathbf{x};\mathbf{d}),\\
&\mathbf{f}^{(1)}_{\mathbf{x},\mathbf{M}}:\mathbb{R}^n\to\mathbb{R}^m:\mathbf{d}\mapsto[\mathbf{f}^{(0)}_{\mathbf{x},\mathbf{M}}]'(\mathbf{m}_{(1)};\mathbf{d}),\\
&\mathbf{f}^{(2)}_{\mathbf{x},\mathbf{M}}:\mathbb{R}^n\to\mathbb{R}^m:\mathbf{d}\mapsto[\mathbf{f}^{(1)}_{\mathbf{x},\mathbf{M}}]'(\mathbf{m}_{(2)};\mathbf{d}),\\
&\qquad\vdots\\
&\mathbf{f}^{(k)}_{\mathbf{x},\mathbf{M}}:\mathbb{R}^n\to\mathbb{R}^m:\mathbf{d}\mapsto[\mathbf{f}^{(k-1)}_{\mathbf{x},\mathbf{M}}]'(\mathbf{m}_{(k)};\mathbf{d}).
\end{aligned}$$

The function $\mathbf{f}$ is said to be *lexicographically smooth (L-smooth) on* X if it is L-smooth at each point $\mathbf{x}\in X$. The class of L-smooth functions is closed under composition, and includes all C^1 functions, convex functions [8], and PC^1 functions [7] in the sense of Scholtes [9]. Given any nonsingular matrix $\mathbf{M}\in\mathbb{R}^{n\times n}$ and $\mathbf{f}:X\to\mathbb{R}^m$ L-smooth at $\mathbf{x}\in X$, the mapping $\mathbf{f}^{(n)}_{\mathbf{x},\mathbf{M}}:\mathbb{R}^n\to\mathbb{R}^m$ is linear and the *lexicographic (L-)derivative* of $\mathbf{f}$ at $\mathbf{x}$ in the directions $\mathbf{M}$ is

$$\mathbf{J}_{\mathrm{L}}\mathbf{f}(\mathbf{x};\mathbf{M}):=\mathbf{J}\mathbf{f}^{(n)}_{\mathbf{x},\mathbf{M}}(\mathbf{0}_n)\in\mathbb{R}^{m\times n}.$$

The *lexicographic subdifferential* of $\mathbf{f}$ at $\mathbf{x}$ is defined as

$$\partial_{\mathrm{L}}\mathbf{f}(\mathbf{x}):=\{\mathbf{J}_{\mathrm{L}}\mathbf{f}(\mathbf{x};\mathbf{N}):\mathbf{N}\in\mathbb{R}^{n\times n},\det\mathbf{N}\neq 0\}.$$

If $\mathbf{f}$ is differentiable at $\mathbf{x}$ then $\partial_{\mathrm{L}}\mathbf{f}(\mathbf{x})=\{\mathbf{J}\mathbf{f}(\mathbf{x})\}$ and if $m=1$ then $\partial_{\mathrm{L}}f(\mathbf{x})\subset\partial f(\mathbf{x})$.

The lexicographic directional derivative was introduced by Khan and Barton [7]: given any $k\in\mathbb{N}$, any $\mathbf{M}:=[\mathbf{m}_{(1)}\ \cdots\ \mathbf{m}_{(k)}]\in\mathbb{R}^{n\times k}$, and $\mathbf{f}:X\to\mathbb{R}^m$ L-smooth at $\mathbf{x}\in X$, the *lexicographic directional (LD-)derivative* of $\mathbf{f}$ at $\mathbf{x}$ in the directions $\mathbf{M}$ is defined as

$$\mathbf{f}'(\mathbf{x};\mathbf{M}):=\left[\mathbf{f}^{(0)}_{\mathbf{x},\mathbf{M}}(\mathbf{m}_{(1)})\quad \mathbf{f}^{(1)}_{\mathbf{x},\mathbf{M}}(\mathbf{m}_{(2)})\quad\cdots\quad \mathbf{f}^{(k-1)}_{\mathbf{x},\mathbf{M}}(\mathbf{m}_{(k)})\right].$$

Note that $\mathbf{f}'(\mathbf{x};\mathbf{M})$ is uniquely defined for all $\mathbf{M}\in\mathbb{R}^{n\times k}$ and all $k\in\mathbb{N}$. The LD-derivative adopts its name because if $\mathbf{M}$ is square and nonsingular then $\mathbf{f}'(\mathbf{x};\mathbf{M})=\mathbf{J}_{\mathrm{L}}\mathbf{f}(\mathbf{x};\mathbf{M})\mathbf{M}$. If $\mathbf{f}$ is differentiable at $\mathbf{x}$ then $\mathbf{f}'(\mathbf{x};\mathbf{M})=\mathbf{J}\mathbf{f}(\mathbf{x})\mathbf{M}$. If $\mathbf{M}$ has one column, the LD-derivative is equivalent to the directional derivative. Unlike the generalized Jacobian, the LD-derivative obeys a strict chain rule [7].

Theorem 2.2. Let $X\subset\mathbb{R}^n$, $Y\subset\mathbb{R}^m$ be open and $\mathbf{h}:X\to Y$ and $\mathbf{g}:Y\to\mathbb{R}^q$ be locally Lipschitz functions on X and Y, respectively. Let $\mathbf{h}$ and $\mathbf{g}$ be L-smooth at $\mathbf{x}\in X$ and $\mathbf{h}(\mathbf{x})\in Y$, respectively. Then the composition $\mathbf{g}\circ\mathbf{h}$ is L-smooth at $\mathbf{x}$; for any $k\in\mathbb{N}$ and any $\mathbf{M}\in\mathbb{R}^{n\times k}$, the chain rule for LD-derivatives is given as:

$$[\mathbf{g}\circ\mathbf{h}]'(\mathbf{x};\mathbf{M})=\mathbf{g}'(\mathbf{h}(\mathbf{x});\mathbf{h}'(\mathbf{x};\mathbf{M})).\tag{1}$$

Theorem 2.2 reduces to Nesterov's chain rule (Theorem 5 in [8]) when the matrix $\mathbf{M}$ is square and nonsingular, and reduces to the classical chain rule when $\mathbf{g}$ and $\mathbf{h}$ are both differentiable. Significantly, the strict chain rule of Theorem 2.2 allows for the development of a vector forward mode of automatic differentiation to calculate LD-derivatives [7].

Remark 2.3. Given an open set $X\subset\mathbb{R}^n$ and $\mathbf{f}:X\to\mathbb{R}^m$ that is L-smooth at $\mathbf{x}\in X$, $\partial_{\mathrm{L}}\mathbf{f}(\mathbf{x})\subset\partial_{\mathrm{P}}\mathbf{f}(\mathbf{x})$ [10]. If $\mathbf{f}$ is PC^1 at $\mathbf{x}$ then $\mathbf{f}$ is L-smooth at $\mathbf{x}$ and $\partial_{\mathrm{L}}\mathbf{f}(\mathbf{x})\subset\partial_{\mathrm{B}}\mathbf{f}(\mathbf{x})$ [7]. Prompted by these relations and the discussions in Remark 2.1 on the usefulness of elements of the plenary Jacobian, obtaining an element of $\partial_{\mathrm{L}}\mathbf{f}(\mathbf{x})$ is therefore just as useful as an element of the Clarke Jacobian in a variety of applications, and can be furnished via computing $\mathbf{f}'(\mathbf{x};\mathbf{M})$ for a square and nonsingular matrix $\mathbf{M}$ and solving the linear equation system $\mathbf{f}'(\mathbf{x};\mathbf{M})=\mathbf{J}_{\mathrm{L}}\mathbf{f}(\mathbf{x};\mathbf{M})\mathbf{M}$ [7].

2.4. Generalized Derivatives of Ordinary Differential Equations

Khan and Barton found LD-derivatives of ODEs in Theorem 4.2 in [10], which is restated here for parametric ODEs whose right-hand side functions depend explicitly on parameters by virtue of its proof and the remarks following Example 4.2 in [10].

Theorem 2.4. Let $n_p, n_x \in \mathbb{N}$, $n_t = 1$, $D \subset \mathbb{R}^{n_t} \times \mathbb{R}^{n_p} \times \mathbb{R}^{n_x}$ be open and connected, $t_0, t_f \in \pi_t D$ satisfy $t_0 < t_f$, and $Z_\mathbf{f}$ be a zero-measure subset of $[t_0, t_f]$. Let $\mathbf{f} : D \to \mathbb{R}^{n_x}$ and $\mathbf{f}_0 : \pi_p D \to \pi_x D$. Assume that the following conditions are satisfied:

(i) $\mathbf{f}(\cdot, \mathbf{p}, \boldsymbol{\eta})$ is measurable on $[t_0, t_f]$ for each $(\mathbf{p}, \boldsymbol{\eta}) \in \pi_{p,x} D$;

(ii) $\mathbf{f}(t, \cdot, \cdot)$ is L-smooth on $\pi_{p,x}(D; t)$ for each $t \in [t_0, t_f] \setminus Z_\mathbf{f}$;

(iii) there exist Lebesgue integrable functions $k_\mathbf{f}, m_\mathbf{f} : [t_0, t_f] \to \mathbb{R}_+ \cup \{+\infty\}$ for which:

  (a) $\|\mathbf{f}(t, \mathbf{p}, \boldsymbol{\eta})\| \leq m_\mathbf{f}(t), \quad \forall t \in [t_0, t_f], \quad \forall (\mathbf{p}, \boldsymbol{\eta}) \in \pi_{p,x}(D; t)$;

  (b) $\|\mathbf{f}(t, \mathbf{p}_1, \boldsymbol{\eta}_1) - \mathbf{f}(t, \mathbf{p}_2, \boldsymbol{\eta}_2)\| \leq k_\mathbf{f}(t)\|(\mathbf{p}_1, \boldsymbol{\eta}_1) - (\mathbf{p}_2, \boldsymbol{\eta}_2)\|, \quad \forall t \in [t_0, t_f]$, $\forall (\mathbf{p}_1, \boldsymbol{\eta}_1), (\mathbf{p}_2, \boldsymbol{\eta}_2) \in \pi_{p,x}(D; t)$;

(iv) $\mathbf{f}_0$ is L-smooth on $\pi_p D$;

(v) for some $\mathbf{p}_0 \in \pi_p D$, there exists a solution $\mathbf{x}(\cdot, \mathbf{p}_0)$ of the following parametric ODE system at $\mathbf{p} := \mathbf{p}_0$:

$$\begin{aligned} \dot{\mathbf{x}}(t, \mathbf{p}) &= \mathbf{f}(t, \mathbf{x}(t, \mathbf{p})), \quad \text{a.e. } t \in [t_0, t_f], \\ \mathbf{x}(t_0, \mathbf{p}) &= \mathbf{f}_0(\mathbf{p}), \end{aligned}$$

which satisfies $\{(t, \mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0)) : t \in [t_0, t_f]\} \subset D$. Then, for each $t \in [t_0, t_f]$, the mapping $\mathbf{x}_t \equiv \mathbf{x}(t, \cdot)$ is Lipschitz continuous on a neighborhood of $\mathbf{p}_0$, with a Lipschitz constant that is independent of t. Moreover, $\mathbf{x}_t$ is L-smooth at $\mathbf{p}_0$; for any $k \in \mathbb{N}$ and any $\mathbf{M} \in \mathbb{R}^{n_p \times k}$, the LD-derivative mapping $\widetilde{\mathbf{X}} : [t_0, t_f] \to \mathbb{R}^{n_x \times k} : t \mapsto [\mathbf{x}_t]'(\mathbf{p}_0; \mathbf{M})$ is the unique solution on $[t_0, t_f]$ of the following ODE system:

$$\begin{aligned} \dot{\mathbf{X}}(t) &= [\mathbf{f}_{t \setminus Z_\mathbf{f}}]'(\mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0); (\mathbf{M}, \mathbf{X}(t))), \\ \mathbf{X}(t_0) &= [\mathbf{f}_0]'(\mathbf{p}_0; \mathbf{M}). \end{aligned} \tag{2}$$

Remark 2.5. The right-hand side function $(t, \mathbf{A}) \mapsto [\mathbf{f}_{t \setminus Z_\mathbf{f}}]'(\mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0); (\mathbf{M}, \mathbf{A}))$ in (2) is measurable with respect to t but not necessarily continuous with respect to $\mathbf{A}$ at almost every $t \in [t_0, t_f]$ (see Example 4.1 in [10]). However, the columns of (2) can be decoupled to yield a sequence of k Carathéodory ODEs (Corollary 4.2 in [10]). Consequently, the k columns of the matrix-valued function $t \mapsto [\mathbf{x}_t]'(\mathbf{p}_0; \mathbf{M})$ are absolutely continuous vector-valued functions mapping $[t_0, t_f]$ to $\mathbb{R}^{n_x}$.

2.5. Lexicographic Smoothness of Extended Implicit Functions

Clarke provided local inverse and implicit function theorems for locally Lipschitz continuous functions [18]; a Lipschitzian function has a local inverse near a point if its Clarke Jacobian is of maximal rank at said point [18]. Levy and Mordukhovich [38] derived an implicit function theorem for coderivatives, and, extending the results of Scholtes [9, Theorem 3.2.3] concerning directional derivative information, Khan and Barton [11] established results on the lexicographic smoothness of local inverse and implicit functions and their corresponding LD-derivatives. For congruence with the present article, the L-smooth implicit function result in [11] is restated with a stricter sufficient condition concerning projections of Clarke Jacobians (see the discussion following Theorem 2 in [11]).

Theorem 2.6. Let $W \subset \mathbb{R}^n \times \mathbb{R}^m$ be open and $\mathbf{g} : W \to \mathbb{R}^m$ be L-smooth at $(\mathbf{x}^*, \mathbf{y}^*) \in W$. Suppose that $\mathbf{g}(\mathbf{x}^*, \mathbf{y}^*) = \mathbf{0}_m$ and $\pi_2 \partial \mathbf{g}(\mathbf{x}^*, \mathbf{y}^*)$ is of maximal rank. Then there exist neighborhoods $N(\mathbf{x}^*) \subset \pi_x W$ and $N(\mathbf{x}^*, \mathbf{y}^*) \subset W$ of $\mathbf{x}^*$ and $(\mathbf{x}^*, \mathbf{y}^*)$, respectively, and a function $\mathbf{r} : N(\mathbf{x}^*) \to \mathbb{R}^m$ that is Lipschitz continuous on $N(\mathbf{x}^*)$ such that, for each $\mathbf{x} \in N(\mathbf{x}^*)$, $(\mathbf{x}, \mathbf{r}(\mathbf{x}))$ is the unique vector in $N(\mathbf{x}^*, \mathbf{y}^*)$ satisfying $\mathbf{g}(\mathbf{x}, \mathbf{r}(\mathbf{x})) = \mathbf{0}_m$. Moreover, $\mathbf{r}$ is L-smooth at $\mathbf{x}^*$; for any $k \in \mathbb{N}$ and any $\mathbf{M} \in \mathbb{R}^{n \times k}$, the LD-derivative $\mathbf{r}'(\mathbf{x}^*; \mathbf{M})$ is the unique solution $\mathbf{N} \in \mathbb{R}^{m \times k}$ of the equation system

$$\mathbf{g}'(\mathbf{x}^*, \mathbf{y}^*; (\mathbf{M}, \mathbf{N})) = \mathbf{0}_{m \times k}. \tag{3}$$

In [35], an extended implicit function theorem was provided for locally Lipschitz continuous functions. The L-smoothness of such an extended implicit function is detailed in the next result.

Theorem 2.7. Let $W \subset \mathbb{R}^n \times \mathbb{R}^m$ be open and $\mathbf{g} : W \to \mathbb{R}^m$ be L-smooth on W. Let $\Omega \subset W$ be a compact set such that each point $\mathbf{x} \in \pi_x\Omega$ is the projection of only one point $(\mathbf{x}, \mathbf{y}) \in \Omega$, $\pi_2\partial\mathbf{g}(\mathbf{x}, \mathbf{y})$ is of maximal rank for each $(\mathbf{x}, \mathbf{y}) \in \Omega$, and $\mathbf{g}(\Omega) = \{\mathbf{0}_m\}$. Then there exist $\delta, \rho > 0$ and a function $\mathbf{r} : B_\delta(\pi_x\Omega) \subset \pi_x W \to \mathbb{R}^m$ that is Lipschitz continuous and L-smooth on $B_\delta(\pi_x\Omega)$ such that $\pi_2\partial\mathbf{g}(\mathbf{x}, \mathbf{y})$ is of maximal rank for all $(\mathbf{x}, \mathbf{y}) \in B_\rho(\Omega) \subset W$ and, for each $\mathbf{x} \in B_\delta(\pi_x\Omega)$, $(\mathbf{x}, \mathbf{r}(\mathbf{x}))$ is the unique vector in $B_\rho(\Omega)$ satisfying $\mathbf{g}(\mathbf{x}, \mathbf{r}(\mathbf{x})) = \mathbf{0}_m$. Moreover, for any $\mathbf{x} \in B_\delta(\pi_x\Omega)$, any $k \in \mathbb{N}$, and any $\mathbf{M} \in \mathbb{R}^{n\times k}$, $\mathbf{r}'(\mathbf{x}; \mathbf{M})$ is the unique solution $\mathbf{N} \in \mathbb{R}^{m\times k}$ of the equation system

$$\mathbf{g}'(\mathbf{x}, \mathbf{r}(\mathbf{x}); (\mathbf{M}, \mathbf{N})) = \mathbf{0}_{m\times k}. \tag{4}$$

Proof. By Theorem 3.6 in [35], there exist $\delta_1, \rho_1 > 0$ and a function

$$\mathbf{r}_1 : B_{\delta_1}(\pi_x\Omega) \subset \pi_x W \to \mathbb{R}^m$$

that is Lipschitz continuous on $B_{\delta_1}(\pi_x\Omega)$ such that $\pi_2\partial\mathbf{g}(\mathbf{x}, \mathbf{y})$ is of maximal rank for all $(\mathbf{x}, \mathbf{y}) \in B_{\rho_1}(\Omega) \subset W$ and, for each $\mathbf{x} \in B_{\delta_1}(\pi_x\Omega)$, $(\mathbf{x}, \mathbf{r}_1(\mathbf{x}))$ is the unique vector in $B_{\rho_1}(\Omega)$ satisfying $\mathbf{g}(\mathbf{x}, \mathbf{r}_1(\mathbf{x})) = \mathbf{0}_m$.

Let

$$\widetilde{\Omega} := \left\{(\mathbf{x}, \mathbf{r}_1(\mathbf{x})) : \mathbf{x} \in \bar{B}_{0.5\delta_1}(\pi_x\Omega)\right\} \subset B_{\rho_1}(\Omega) \subset W,$$

which is a compact set as the image of compact set under continuous mapping. Furthermore, each point

$$\mathbf{x} \in \pi_x\widetilde{\Omega} = \bar{B}_{0.5\delta_1}(\pi_x\Omega) \subset B_{\delta_1}(\pi_x\Omega)$$

is the projection of only one point in $\widetilde{\Omega}$ (namely, $(\mathbf{x}, \mathbf{r}_1(\mathbf{x}))$). Moreover, $\pi_2\mathbf{g}(\mathbf{x}, \mathbf{y})$ is of maximal rank for all $(\mathbf{x}, \mathbf{y}) \in \widetilde{\Omega} \subset B_{\rho_1}(\Omega)$ and $\mathbf{g}(\widetilde{\Omega}) = \{\mathbf{0}_m\}$. Theorem 3.6 in [35] can therefore be applied once more to yield the existence of $\delta_2, \rho_2 > 0$ and a function

$$\mathbf{r}_2 : B_{\delta_2}(\pi_x\widetilde{\Omega}) \subset \pi_x W \to \mathbb{R}^m$$

that is Lipschitz continuous on $B_{\delta_2}(\pi_x\widetilde{\Omega})$ such that $\pi_2\partial\mathbf{g}(\mathbf{x}, \mathbf{y})$ is of maximal rank for all $(\mathbf{x}, \mathbf{y}) \in B_{\rho_2}(\widetilde{\Omega}) \subset W$ and, for each $\mathbf{x} \in B_{\delta_2}(\pi_x\widetilde{\Omega})$, $(\mathbf{x}, \mathbf{r}_2(\mathbf{x}))$ is the unique vector in $B_{\rho_2}(\widetilde{\Omega})$ satisfying $\mathbf{g}(\mathbf{x}, \mathbf{r}_2(\mathbf{x})) = \mathbf{0}_m$.

Choose any $\hat{\mathbf{x}} \in \pi_x\widetilde{\Omega}$. By virtue of the proof of Theorem 3.6 in [35], there exist a neighborhood $N(\hat{\mathbf{x}}) \subset \pi_x W$ of $\hat{\mathbf{x}}$ and a Lipschitz continuous function

$$\mathbf{r}_{\hat{\mathbf{x}}} : N(\hat{\mathbf{x}}) \to \mathbb{R}^m$$

such that $\mathbf{g}(\hat{\mathbf{x}}, \hat{\mathbf{y}}) = \mathbf{0}_m$, where $\hat{\mathbf{y}} := \mathbf{r}_{\hat{\mathbf{x}}}(\hat{\mathbf{x}})$, and $\pi_2\partial\mathbf{g}(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is of maximal rank. Moreover, $\mathbf{r}_{\hat{\mathbf{x}}} = \mathbf{r}_2$ on $N(\hat{\mathbf{x}}) \cap \pi_x\widetilde{\Omega}$. By Theorem 2.6, $\mathbf{r}_{\hat{\mathbf{x}}}$ is L-smooth at $\hat{\mathbf{x}}$; for any $k \in \mathbb{N}$ and any $\mathbf{M} \in \mathbb{R}^{n\times k}$, $[\mathbf{r}_{\hat{\mathbf{x}}}]'(\hat{\mathbf{x}}; \mathbf{M}) = [\mathbf{r}_2]'(\hat{\mathbf{x}}; \mathbf{M})$ is the unique solution $\mathbf{N} \in \mathbb{R}^{m\times k}$ of the equation system

$$\mathbf{0}_{m\times k} = \mathbf{g}'(\hat{\mathbf{x}}, \hat{\mathbf{y}}; (\mathbf{M}, \mathbf{N})) = \mathbf{g}'(\hat{\mathbf{x}}, \mathbf{r}_2(\hat{\mathbf{x}}); (\mathbf{M}, \mathbf{N})).$$

Let $\delta := 0.5\delta_1$, $\rho := \rho_1$, and

$$\mathbf{r} : B_\delta(\pi_x\Omega) \subset \pi_x W \to \mathbb{R}^m : \boldsymbol{\eta} \mapsto \mathbf{r}_2(\boldsymbol{\eta}).$$

$\mathbf{r}$ is Lipschitz continuous on $B_\delta(\pi_x\Omega) \subset \pi_x\widetilde{\Omega} \subset B_{\delta_2}(\pi_x\widetilde{\Omega})$ and $\pi_2\partial\mathbf{g}(\mathbf{x}, \mathbf{y})$ is of maximal rank for all $(\mathbf{x}, \mathbf{y}) \in B_\rho(\widetilde{\Omega}) = B_{\rho_1}(\widetilde{\Omega}) \subset W$. By uniqueness, $\mathbf{r}_1 = \mathbf{r}_2$ on $B_{\delta_1}(\pi_x\Omega) \cap B_{\delta_2}(\pi_x\widetilde{\Omega}) \supset B_\delta(\pi_x\Omega)$; for each $\mathbf{x} \in B_\delta(\pi_x\Omega)$,

$$(\mathbf{x}, \mathbf{r}(\mathbf{x})) = (\mathbf{x}, \mathbf{r}_1(\mathbf{x})) = (\mathbf{x}, \mathbf{r}_2(\mathbf{x}))$$

is the unique vector in $B_{\rho_1}(\widetilde{\Omega})$ satisfying $\mathbf{g}(\mathbf{x}, \mathbf{r}(\mathbf{x})) = \mathbf{0}_m$. $\mathbf{r}$ is L-smooth on $B_\delta(\pi_x\Omega) \subset \pi_x\widetilde{\Omega}$; for any $\mathbf{x} \in B_\delta(\pi_x\Omega)$, any $k \in \mathbb{N}$, and any $\mathbf{M} \in \mathbb{R}^{n\times k}$, $\mathbf{r}'(\mathbf{x}; \mathbf{M}) = [\mathbf{r}_2]'(\mathbf{x}; \mathbf{M})$ is the unique solution $\mathbf{N} \in \mathbb{R}^{m\times k}$ of the equation system

$$\mathbf{0}_{m\times k} = \mathbf{g}'(\mathbf{x}, \mathbf{r}_2(\mathbf{x}); (\mathbf{M}, \mathbf{N})) = \mathbf{g}'(\mathbf{x}, \mathbf{r}(\mathbf{x}); (\mathbf{M}, \mathbf{N})).$$

□

Remark 2.8. The implicit function $\mathbf{r}$ outlined in the statement of Theorem 2.7 is L-smooth on its open domain $B_\delta(\pi_x\Omega) \supset \pi_x\Omega$, which is needed for the higher-order directional derivatives outlined earlier to be well-defined and is essential for the analysis to follow. The fact that $\mathbf{r}$ is also Lipschitz continuous on $B_\delta(\pi_x\Omega)$ is not immediately implied by its L-smoothness. Moreover, when Ω is a singleton, Theorem 2.6 is recovered.

The next lemma is derived for later application of Theorem 2.7 to a nonsmooth differential-algebraic equation system.

Lemma 2.9. Let $W \subset \mathbb{R}^n \times \mathbb{R}^m$ be open and $\mathbf{g} : W \to \mathbb{R}^q$ be locally Lipschitz continuous on W and L-smooth at $(\mathbf{x}, \mathbf{y}) \in W$. Given any $k \in \mathbb{N}$ and any matrix $\mathbf{M} := [\mathbf{m}_{(1)} \ \cdots \ \mathbf{m}_{(k)}] \in \mathbb{R}^{m\times k}$,

$$[\mathbf{g_x}]'(\mathbf{y}; \mathbf{M}) = \mathbf{g}'(\mathbf{x}, \mathbf{y}; (\mathbf{0}_{n\times k}, \mathbf{M})),$$

where $\mathbf{g_x} \equiv \mathbf{g}(\mathbf{x}, \cdot)$.

Proof. Let $\bar{\mathbf{M}} := (\mathbf{0}_{n\times k}, \mathbf{M})$ and $\mathbf{w} := (\mathbf{x}, \mathbf{y})$. The LD-derivative of $\mathbf{g}$ at $\mathbf{w}$ in the directions $\bar{\mathbf{M}} := [\bar{\mathbf{m}}_{(1)} \ \cdots \ \bar{\mathbf{m}}_{(k)}]$ is given by

$$\begin{aligned}
\mathbf{g}'(\mathbf{w}; \bar{\mathbf{M}}) &= \left[\mathbf{g}^{(0)}_{\mathbf{w},\bar{\mathbf{M}}}(\bar{\mathbf{m}}_{(1)}) \quad \mathbf{g}^{(1)}_{\mathbf{w},\bar{\mathbf{M}}}(\bar{\mathbf{m}}_{(2)}) \quad \cdots \quad \mathbf{g}^{(k-1)}_{\mathbf{w},\bar{\mathbf{M}}}(\bar{\mathbf{m}}_{(k)})\right], \\
&= \left[\mathbf{g}^{(0)}_{(\mathbf{x},\mathbf{y}),(\mathbf{0}_{n\times k},\mathbf{M})}\left(\begin{bmatrix}\mathbf{0}_n \\ \mathbf{m}_{(1)}\end{bmatrix}\right) \quad \cdots \quad \mathbf{g}^{(k-1)}_{(\mathbf{x},\mathbf{y}),(\mathbf{0}_{n\times k},\mathbf{M})}\left(\begin{bmatrix}\mathbf{0}_n \\ \mathbf{m}_{(k)}\end{bmatrix}\right)\right], \\
&= \mathbf{g}'(\mathbf{x}, \mathbf{y}; (\mathbf{0}_{n\times k}, \mathbf{M})).
\end{aligned}$$

It will be shown by induction that

$$\mathbf{g}^{(i)}_{\mathbf{w},\bar{\mathbf{M}}}\left(\begin{bmatrix}\mathbf{0}_n \\ \mathbf{d}\end{bmatrix}\right) = [\mathbf{g}(\mathbf{x}, \cdot)]^{(i)}_{\mathbf{y},\mathbf{M}}(\mathbf{d}), \quad \forall \mathbf{d} \in \mathbb{R}^m, \forall i \in \{0, 1, \ldots, k-1\},$$

Choose any $\mathbf{d} \in \mathbb{R}^m$ and let $\bar{\mathbf{d}} := \begin{bmatrix}\mathbf{0}_n \\ \mathbf{d}\end{bmatrix}$, then

$$\begin{aligned}
\mathbf{g}^{(0)}_{\mathbf{w},\bar{\mathbf{M}}}(\bar{\mathbf{d}}) &= \lim_{\alpha\downarrow 0} \alpha^{-1}(\mathbf{g}(\mathbf{x}, \mathbf{y} + \alpha\mathbf{d}) - \mathbf{g}(\mathbf{x}, \mathbf{y})), \\
&= \lim_{\alpha\downarrow 0} \alpha^{-1}([\mathbf{g}(\mathbf{x}, \cdot)](\mathbf{y} + \alpha\mathbf{d}) - [\mathbf{g}(\mathbf{x}, \cdot)](\mathbf{y})), \\
&= [\mathbf{g}(\mathbf{x}, \cdot)]^{(0)}_{\mathbf{y},\mathbf{M}}(\mathbf{d}).
\end{aligned}$$

Assume that the claim is true for $i := j \in \{0, 1, \ldots, k-2\}$. Then, for any $\mathbf{d} \in \mathbb{R}^m$,

$$\begin{aligned}
\mathbf{g}^{(j+1)}_{\mathbf{w},\bar{\mathbf{M}}}(\bar{\mathbf{d}}) &= [\mathbf{g}^{(j)}_{\mathbf{w},\bar{\mathbf{M}}}]'(\bar{\mathbf{m}}_{(j+1)}; \bar{\mathbf{d}}), \\
&= [\mathbf{g}^{(j)}_{\mathbf{w},\bar{\mathbf{M}}}]'\left(\begin{bmatrix}\mathbf{0}_n \\ \mathbf{m}_{(j+1)}\end{bmatrix}; \begin{bmatrix}\mathbf{0}_n \\ \mathbf{d}\end{bmatrix}\right), \\
&= \lim_{\alpha\downarrow 0} \alpha^{-1}(\mathbf{g}^{(j)}_{\mathbf{w},\bar{\mathbf{M}}}(\mathbf{0}_n, \mathbf{m}_{(j+1)} + \alpha\mathbf{d}) - \mathbf{g}^{(j)}_{\mathbf{w},\bar{\mathbf{M}}}(\mathbf{0}_n, \mathbf{m}_{(j+1)})), \\
&= \lim_{\alpha\downarrow 0} \alpha^{-1}([\mathbf{g}(\mathbf{x}, \cdot)]^{(j)}_{\mathbf{y},\mathbf{M}}(\mathbf{m}_{(j+1)} + \alpha\mathbf{d}) - [\mathbf{g}(\mathbf{x}, \cdot)]^{(j)}_{\mathbf{y},\mathbf{M}}(\mathbf{m}_{(j+1)})), \\
&= [[\mathbf{g}(\mathbf{x}, \cdot)]^{(j)}_{\mathbf{y},\mathbf{M}}]'(\mathbf{m}_{(j+1)}; \mathbf{d}), \\
&= [\mathbf{g}(\mathbf{x}, \cdot)]^{(j+1)}_{\mathbf{y},\mathbf{M}}(\mathbf{d}),
\end{aligned}$$

and therefore the claim holds. It follows that

$$\begin{aligned}
[\mathbf{g_x}]'(\mathbf{y}; \mathbf{M}) &= \left[[\mathbf{g}(\mathbf{x}, \cdot)]^{(0)}_{\mathbf{y},\mathbf{M}}(\mathbf{m}_{(1)}) \quad [\mathbf{g}(\mathbf{x}, \cdot)]^{(1)}_{\mathbf{y},\mathbf{M}}(\mathbf{m}_{(2)}) \quad \cdots \quad [\mathbf{g}(\mathbf{x}, \cdot)]^{(k-1)}_{\mathbf{y},\mathbf{M}}(\mathbf{m}_{(k)})\right], \\
&= \mathbf{g}'(\mathbf{x}, \mathbf{y}; (\mathbf{0}_{n\times k}, \mathbf{M})).
\end{aligned}$$

□

3. Forward Sensitivity Functions for Nonsmooth Differential-Algebraic Equations

Let $n_p, n_x, n_y \in \mathbb{N}$. Let $D_t \subset \mathbb{R}$, $D_p \subset \mathbb{R}^{n_p}$, $D_y \subset \mathbb{R}^{n_y}$, and $D_x \subset \mathbb{R}^{n_x}$ be open and connected sets. Let $D := D_t \times D_p \times D_x \times D_y$, $\mathbf{f} : D \to \mathbb{R}^{n_x}$, $\mathbf{g} : D \to \mathbb{R}^{n_y}$, and $\mathbf{f}_0 : D_p \to D_x$. Given $t_0 \in D_t$, consider the following initial-value problem (IVP) in semi-explicit DAEs:

$$\dot{\mathbf{x}}(t, \mathbf{p}) = \mathbf{f}(t, \mathbf{p}, \mathbf{x}(t, \mathbf{p}), \mathbf{y}(t, \mathbf{p})), \tag{5a}$$

$$\mathbf{0}_{n_y} = \mathbf{g}(t, \mathbf{p}, \mathbf{x}(t, \mathbf{p}), \mathbf{y}(t, \mathbf{p})), \tag{5b}$$

$$\mathbf{x}(t_0, \mathbf{p}) = \mathbf{f}_0(\mathbf{p}), \tag{5c}$$

where t is the independent variable and $\mathbf{p} \in D_p$ is a vector of the problem parameters. The following assumption is made regarding the right-hand side functions in (5).

Assumption 3.1. Let $t_f \in D_t$ satisfy $t_0 < t_f$ and $Z_{\mathbf{f}}$ be a zero-measure subset of $[t_0, t_f]$. Suppose that the following conditions hold:

(i) $\mathbf{f}(\cdot, \mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y)$ is measurable on $[t_0, t_f]$ for each $(\mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y) \in D_p \times D_x \times D_y$;

(ii) $\mathbf{f}(t, \cdot, \cdot, \cdot)$ is L-smooth on $D_p \times D_x \times D_y$ for each $t \in [t_0, t_f] \setminus Z_{\mathbf{f}}$;

(iii) there exist Lebesgue integrable functions $k_{\mathbf{f}}, m_{\mathbf{f}} : [t_0, t_f] \to \mathbb{R}_+ \cup \{+\infty\}$ for which:

  (a) $\|\mathbf{f}(t, \mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y)\| \leq m_{\mathbf{f}}(t), \quad \forall t \in [t_0, t_f], \quad \forall (\mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y) \in D_p \times D_x \times D_y$;

  (b) $\|\mathbf{f}(t, \mathbf{p}_1, \boldsymbol{\eta}_{x_1}, \boldsymbol{\eta}_{y_1}) - \mathbf{f}(t, \mathbf{p}_2, \boldsymbol{\eta}_{x_2}, \boldsymbol{\eta}_{y_2})\| \leq k_{\mathbf{f}}(t)\|(\mathbf{p}_1, \boldsymbol{\eta}_{x_1}, \boldsymbol{\eta}_{y_1}) - (\mathbf{p}_2, \boldsymbol{\eta}_{x_2}, \boldsymbol{\eta}_{y_2})\|$, $\forall t \in [t_0, t_f], \quad \forall (\mathbf{p}_1, \boldsymbol{\eta}_{x_1}, \boldsymbol{\eta}_{y_1}), (\mathbf{p}_2, \boldsymbol{\eta}_{x_2}, \boldsymbol{\eta}_{y_2}) \in D_p \times D_x \times D_y$;

(iv) $\mathbf{g}$ and $\mathbf{f}_0$ are L-smooth on D and D_p, respectively.

Notions of consistent initialization, regularity, and solutions of (5) from [35] are reproduced here for the reader's convenience.

Definition 3.2. The *consistency set, initial consistency set, and regularity set of* (5) are given by, respectively,

$$\begin{aligned}
\mathcal{G}_{\mathrm{C}} &:= \{(t, \mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y) \in D : \mathbf{g}(t, \mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y) = \mathbf{0}_{n_y}\}, \\
\mathcal{G}_{\mathrm{C},0} &:= \{(t, \mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y) \in \mathcal{G}_{\mathrm{C}} : t = t_0, \boldsymbol{\eta}_x = \mathbf{f}_0(\mathbf{p})\}, \\
\mathcal{G}_{\mathrm{R}} &:= \{(t, \mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y) \in D : \pi_4 \partial \mathbf{g}(t, \mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y) \text{ is of maximal rank}\}.
\end{aligned}$$

Definition 3.3. Let $T \subset D_t$ be a connected set containing t_0, $P \subset D_p$, and $\Omega_0 \subset \mathcal{G}_{\mathrm{C},0}$. A mapping $\mathbf{z} \equiv (\mathbf{x}, \mathbf{y}) : T \times P \to D_x \times D_y$ is called a *solution of* (5) *on* $T \times P$ *through* Ω_0 if, for each $\mathbf{p} \in P$, $\mathbf{z}(\cdot, \mathbf{p})$ is an absolutely continuous function on T which satisfies (5a) for almost every $t \in T$, (5b) for every $t \in T$, (5c) at $t = t_0$, and

$$\{(t_0, \mathbf{p}, \mathbf{x}(t_0, \mathbf{p}), \mathbf{y}(t_0, \mathbf{p})) : \mathbf{p} \in P\} = \Omega_0.$$

If, in addition,

$$\{(t, \mathbf{p}, \mathbf{x}(t, \mathbf{p}), \mathbf{y}(t, \mathbf{p})) : (t, \mathbf{p}) \in T \times P\} \subset \mathcal{G}_{\mathrm{R}},$$

then $\mathbf{z}$ is called a *regular solution of* (5) *on* $T \times P$ *through* Ω_0.

Definition 3.4. Let $\mathbf{z}$ be a solution of (5) on $T \times P$ through Ω_0. Then $\mathbf{z}$ is said to be *unique* if, given any other solution $\mathbf{z}^*$ of (5) on $T^* \times P^*$ through Ω_0^* satisfying $T \cap T^* \neq \{t_0\}$, $P \cap P^* \neq \emptyset$, and

$$\{(t_0, \mathbf{p}, \mathbf{z}(t_0, \mathbf{p})) : \mathbf{p} \in P \cap P^*\} = \{(t_0, \mathbf{p}, \mathbf{z}^*(t_0, \mathbf{p})) : \mathbf{p} \in P \cap P^*\},$$

$\mathbf{z}(t, \mathbf{p}) = \mathbf{z}^*(t, \mathbf{p})$ for all $(t, \mathbf{p}) \in (T \cap T^*) \times (P \cap P^*)$.

A generalization of the notion that (5) has differential index equal to one (see [14, 22]) for all $(t, \mathbf{p}) \in T \times P$ is implied by regularity. The following assumption regarding the existence of a solution is made.

Assumption 3.5. Suppose that for some $(\mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0) \in D_p \times D_x \times D_y$, there exists a regular solution $\mathbf{z}$ of (5) on $[t_0, t_f] \times \{\mathbf{p}_0\}$ through $\{(t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0)\}$.

Before proceeding to the main result, a result is proved concerning uniqueness and parametric dependence of solutions of (5), as well as its equivalence to a Carathéodory ODE system via an extended implicit function.

Proposition 3.6. Let Assumptions 3.1 and 3.5 hold. Then there exists a neighborhood $N(\mathbf{p}_0) \subset D_p$ of $\mathbf{p}_0$, a set $\Omega_0 \subset \mathcal{G}_{C,0}$ containing $(t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0)$, and a unique regular solution $\mathbf{z}$ of (5) on $[t_0, t_f] \times N(\mathbf{p}_0)$ through Ω_0. Furthermore, there exist $\delta, \rho > 0$ and a function

$$\mathbf{r} : B_\delta(\{(t, \mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0)) : t \in [t_0, t_f]\}) \subset D_t \times D_p \times D_x \to \mathbb{R}^{n_y},$$

that is Lipschitz continuous and L-smooth on its open and connected domain, which satisfy $\mathbf{y}(t, \mathbf{p}) = \mathbf{r}(t, \mathbf{p}, \mathbf{x}(t, \mathbf{p}))$ for all $(t, \mathbf{p}) \in [t_0, t_f] \times N(\mathbf{p}_0)$ and

$$\begin{aligned}
&\{(t, \mathbf{p}, \mathbf{x}(t, \mathbf{p})) : (t, \mathbf{p}) \in [t_0, t_f] \times N(\mathbf{p}_0)\} \subset B_\delta(\{(t, \mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0)) : t \in [t_0, t_f]\}),\\
&\{(t, \mathbf{p}, \mathbf{z}(t, \mathbf{p})) : (t, \mathbf{p}) \in [t_0, t_f] \times N(\mathbf{p}_0)\} \subset B_\rho(\{(t, \mathbf{p}_0, \mathbf{z}(t, \mathbf{p}_0)) : t \in [t_0, t_f]\}) \subset D.
\end{aligned}$$

Proof. Define the following ets:

$$\begin{aligned}
\Lambda &:= \{(t, \mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0)) : t \in [t_0, t_f]\},\\
\Omega &:= \{(t, \mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0), \mathbf{y}(t, \mathbf{p}_0)) : t \in [t_0, t_f]\}.
\end{aligned}$$

Note that the set Ω is compact since it is the image of a compact set under a continuous mapping. $\pi_4 \partial \mathbf{g}(t, \mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y)$ is of maximal rank for all $(t, \mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y) \in \Omega$ by regularity and $\mathbf{g}(\Omega) = \{\mathbf{0}_{n_y}\}$ by consistency. Each point in Λ is the projection of a unique point in Ω by Lemma 3.8 in [35]. By Theorem 2.7, there exist $\delta_1, \rho_1 > 0$ and a function

$$\mathbf{r} : B_{\delta_1}(\Lambda) \subset D_t \times D_p \times D_x \to \mathbb{R}^{n_y}$$

that is Lipschitz continuous and L-smooth on $B_{\delta_1}(\Lambda)$ such that $\pi_4 \partial \mathbf{g}(t, \mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y)$ is of maximal rank for all $(t, \mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y) \in B_{\rho_1}(\Omega) \subset D$ and, for each $(t, \mathbf{p}, \boldsymbol{\eta}_x) \in B_{\delta_1}(\Lambda)$, $(t, \mathbf{p}, \boldsymbol{\eta}_x, \mathbf{r}(t, \mathbf{p}, \boldsymbol{\eta}_x))$ is the unique vector in $B_{\rho_1}(\Omega)$ satisfying $\mathbf{g}(t, \mathbf{p}, \boldsymbol{\eta}_x, \mathbf{r}(t, \mathbf{p}, \boldsymbol{\eta}_x)) = \mathbf{0}_{n_y}$.

By proceeding as in the proof of Theorem 4.34 in [35] using the inherited L-smoothness of the implicit function in place of the Lipschitzian construction, the following conclusions are immediately furnished: there exist $\xi, \beta > 0$ satisfying $\beta < \xi$ and a regular solution $\mathbf{z}$ of (5) on $[t_0, t_f] \times B_\beta(\mathbf{p}_0) \subset D_t \times D_p$ through

$$\Omega_0 := \{(t, \mathbf{p}, \boldsymbol{\eta}_x, \boldsymbol{\eta}_y) : t = t_0, \mathbf{p} \in B_\beta(\mathbf{p}_0), \boldsymbol{\eta}_x = \mathbf{f}_0(\mathbf{p}), \boldsymbol{\eta}_y = \mathbf{r}(t_0, \mathbf{p}, \mathbf{f}_0(\mathbf{p}))\} \subset \mathcal{G}_{C,0}$$

such that $\mathbf{y}(t, \mathbf{p}) = \mathbf{r}(t, \mathbf{p}, \mathbf{x}(t, \mathbf{p}))$ for all $\mathbf{p} \in B_\beta(\mathbf{p}_0)$ and

$$\{(t, \mathbf{p}, \mathbf{x}(t, \mathbf{p}), \mathbf{y}(t, \mathbf{p})) : (t, \mathbf{p}) \in [t_0, t_f] \times B_\beta(\mathbf{p}_0)\} \subset B_{\rho_1}(\Omega).$$

Moreover, the intermediate construction $\mathbf{u}$ in the proof of Theorem 4.34 in [35] satisfies

$$\begin{aligned}
&\{(t, \mathbf{u}(t, \mathbf{c})) : (t, \mathbf{c}) \in [t_0, t_f] \times B_\xi(\mathbf{p}_0, \mathbf{f}_0(\mathbf{p}_0))\} \subset B_{0.5\delta_1}(\Lambda),\\
&\begin{bmatrix} \mathbf{p} \\ \mathbf{x}(t, \mathbf{p}) \end{bmatrix} = \mathbf{u}(t, (\mathbf{p}, \mathbf{f}_0(\mathbf{p}))), \quad \forall (t, \mathbf{p}) \in [t_0, t_f] \times B_\beta(\mathbf{p}_0),
\end{aligned}$$

and $\{(\mathbf{p}, \mathbf{f}_0(\mathbf{p})) : \mathbf{p} \in B_\beta(\mathbf{p}_0)\} \subset B_\xi(\mathbf{p}_0, \mathbf{f}_0(\mathbf{p}_0))$. Thus,

$$\begin{aligned}
&\{(t, \mathbf{p}, \mathbf{x}(t, \mathbf{p})) : (t, \mathbf{p}) \in [t_0, t_f] \times B_\beta(\mathbf{p}_0)\}\\
&= \{(t, \mathbf{u}(t, (\mathbf{p}, \mathbf{f}_0(\mathbf{p}))) : (t, \mathbf{p}) \in [t_0, t_f] \times B_\beta(\mathbf{p}_0)\},\\
&\subset \{(t, \mathbf{u}(t, \mathbf{c})) : (t, \mathbf{c}) \in [t_0, t_f] \times B_\xi(\mathbf{p}_0, \mathbf{f}_0(\mathbf{p}_0))\},\\
&\subset B_{0.5\delta_1}(\Lambda).
\end{aligned}$$

Theorem 4.22 in [35] implies $\mathbf{z}$ is the unique regular solution of (5) on $[t_0, t_f] \times B_\beta(\mathbf{p}_0)$ through Ω_0 and the result holds with $N(\mathbf{p}_0) := B_\beta(\mathbf{p}_0)$, $\delta := 0.5\delta_1$, and $\rho := \rho_1$. □

Using lexicographic differentiation, forward sensitivity functions for (5) are given.

Theorem 3.7. Let Assumptions 3.1 and 3.5 hold. Then, for each $t \in [t_0, t_f]$, the mapping $\mathbf{z}_t \equiv \mathbf{z}(t, \cdot)$ is L-smooth at $\mathbf{p}_0$; for any $k \in \mathbb{N}$ and any $\mathbf{M} \in \mathbb{R}^{n_p \times k}$, the LD-derivative mapping

$$\widetilde{\mathbf{Z}} \equiv (\widetilde{\mathbf{X}}, \widetilde{\mathbf{Y}}) : [t_0, t_f] \to \mathbb{R}^{(n_x+n_y)\times k} : t \mapsto [\mathbf{z}_t]'(\mathbf{p}_0; \mathbf{M})$$

is such that $\widetilde{\mathbf{X}}$ and $\widetilde{\mathbf{Y}}$ are absolutely continuous and Lebesgue integrable on $[t_0, t_f]$, respectively. Furthermore, $\widetilde{\mathbf{Z}}$ uniquely (in the sense of Definition 3.4) satisfies the following DAE system:

$$\begin{aligned} \dot{\mathbf{X}}(t) &= [\mathbf{f}_{t\setminus Z_\mathbf{f}}]'(\mathbf{p}_0, \mathbf{x}(t,\mathbf{p}_0), \mathbf{y}(t,\mathbf{p}_0); (\mathbf{M}, \mathbf{X}(t), \mathbf{Y}(t))), \quad \text{a.e. } t \in [t_0, t_f], \\ \mathbf{0}_{n_y \times k} &= [\mathbf{g}_t]'(\mathbf{p}_0, \mathbf{x}(t,\mathbf{p}_0), \mathbf{y}(t,\mathbf{p}_0); (\mathbf{M}, \mathbf{X}(t), \mathbf{Y}(t))), \qquad \forall t \in [t_0, t_f], \\ \mathbf{X}(t_0) &= [\mathbf{f}_0]'(\mathbf{p}_0; \mathbf{M}), \end{aligned} \tag{6}$$

on $[t_0, t_f]$ through $\{(t_0, \mathbf{X}_0, \mathbf{Y}_0)\}$, where $\mathbf{X}_0 := [\mathbf{f}_0]'(\mathbf{p}_0; \mathbf{M})$ and $\mathbf{Y}_0 \in \mathbb{R}^{n_y \times k}$ is the unique solution of the equation system

$$\mathbf{0}_{n_y \times k} = [\mathbf{g}_{t_0}]'(\mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0; (\mathbf{M}, \mathbf{X}_0, \mathbf{Y}_0)).$$

Proof. Let $\delta, \rho > 0$, $\mathbf{r}$, and $N(\mathbf{p}_0)$ be given as in the statement of Propositions 3.6. Define the sets

$$\begin{aligned} D_\delta &:= B_\delta(\{(t, \mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0)) : t \in [t_0, t_f]\}) \subset D_t \times D_p \times D_x, \\ D_\rho &:= B_\rho(\{(t, \mathbf{p}_0, \mathbf{z}(t, \mathbf{p}_0)) : t \in [t_0, t_f]\}) \subset D, \end{aligned}$$

and the following mappings:

$$\begin{aligned} \mathbf{q} &: D_\delta \to D_p \times D_x \times D_y : (t, \mathbf{p}, \boldsymbol{\eta}_x) \mapsto (\mathbf{p}, \boldsymbol{\eta}_x, \mathbf{r}(t, \mathbf{p}, \boldsymbol{\eta}_x)), \\ \bar{\mathbf{f}} &: D_\delta \to \mathbb{R}^{n_x} : (t, \mathbf{p}, \boldsymbol{\eta}_x) \mapsto \mathbf{f}(t, \mathbf{q}(t, \mathbf{p}, \boldsymbol{\eta}_x)). \end{aligned}$$

For each $(\mathbf{p}, \boldsymbol{\eta}_x) \in \pi_{p,x} D_\delta \subset D_p \times D_x$,

$$\bar{\mathbf{f}}(\cdot, \mathbf{p}, \boldsymbol{\eta}_x) \equiv \mathbf{f}(\cdot, \mathbf{q}(\cdot, \mathbf{p}, \boldsymbol{\eta}_x)) : \pi_t(D_\delta; (\mathbf{p}, \boldsymbol{\eta}_x)) \subset D_t \to \mathbb{R}^{n_x}$$

is measurable on $[t_0, t_f]$ by Lemma 1 in Chapter 1, Section 1 [39] since the mapping $t \mapsto (\mathbf{p}, \boldsymbol{\eta}_x, \mathbf{r}(t, \mathbf{p}, \boldsymbol{\eta}_x))$ is continuous on $\pi_t(D_\delta; (\mathbf{p}, \boldsymbol{\eta}_x)) \supset [t_0, t_f]$,

$$[t_0, t_f] \times \mathbf{q}(D_\delta) \subset [t_0, t_f] \times D_p \times D_x \times D_y, \tag{7}$$

and $\mathbf{f}$ satisfies the Carathéodory conditions (see, e.g., [39]) on $[t_0, t_f] \times D_p \times D_x \times D_y$ by assumption.

L-smoothness of $\bar{\mathbf{f}}(t, \cdot, \cdot)$ is demonstrated as follows: for each $t \in [t_0, t_f]$, the mapping $(\mathbf{p}, \boldsymbol{\eta}_x) \mapsto (\mathbf{p}, \boldsymbol{\eta}_x, \mathbf{r}(t, \mathbf{p}, \boldsymbol{\eta}_x))$ is L-smooth on $\pi_{p,x}(D_\delta; t)$ by L-smoothness of $\mathbf{r}$ on D_δ (and hence $\mathbf{r}_t \equiv \mathbf{r}(t, \cdot, \cdot)$ on $\pi_{p,x}(D_\delta; t)$). Thus, $\mathbf{q}_t \equiv \mathbf{q}(t, \cdot, \cdot)$ is L-smooth on $\pi_{p,x}(D_\delta; t)$. Since

$$\mathbf{q}_t(\pi_{p,x}(D_\delta; t)) \subset \pi_{p,x,y}(D_\rho; t) \subset D_p \times D_x \times D_y, \quad \forall t \in [t_0, t_f] \setminus Z_\mathbf{f},$$

and the composition of L-smooth functions is L-smooth, it follows that

$$\bar{\mathbf{f}}(t, \cdot, \cdot) \equiv \mathbf{f}(t, \mathbf{q}(t, \cdot, \cdot)) : \pi_{p,x}(D_\delta; t) \subset D_p \times D_x \to \mathbb{R}^{n_x}$$

is L-smooth on $\pi_{p,x}(D_\delta; t)$ for each $t \in [t_0, t_f] \setminus Z_\mathbf{f}$.

For any $t \in [t_0, t_f]$ and any $(\mathbf{p}, \boldsymbol{\eta}_x) \in \pi_{p,x}(D_\delta; t)$,

$$\|\bar{\mathbf{f}}(t, \mathbf{p}, \boldsymbol{\eta}_x)\| = \|\mathbf{f}(t, \mathbf{p}, \boldsymbol{\eta}_x, \mathbf{r}(t, \mathbf{p}, \boldsymbol{\eta}_x))\| \le m_\mathbf{f}(t),$$

by (7) and the Carathéodory conditions of $\mathbf{f}$. By Lipschitz continuity of $\mathbf{r}$ on D_δ, there exists $k_\mathbf{r} \ge 0$ such that

$$\|\mathbf{r}(t, \mathbf{p}_1, \boldsymbol{\eta}_{x_1}) - \mathbf{r}(t, \mathbf{p}_2, \boldsymbol{\eta}_{x_2})\| \le k_\mathbf{r} \|(\mathbf{p}_1, \boldsymbol{\eta}_{x_1}) - (\mathbf{p}_2, \boldsymbol{\eta}_{x_2})\|, \tag{8}$$

for any $(t, \mathbf{p}_1, \boldsymbol{\eta}_{x_1}), (t, \mathbf{p}_2, \boldsymbol{\eta}_{x_2}) \in D_\delta$. It follows that

$$\begin{aligned}\|\bar{\mathbf{f}}(t, \mathbf{p}_1, \boldsymbol{\eta}_{x_1}) - \bar{\mathbf{f}}(t, \mathbf{p}_2, \boldsymbol{\eta}_{x_2})\| &= \|\mathbf{f}(t, \mathbf{p}_1, \boldsymbol{\eta}_{x_1}, \mathbf{r}(t, \mathbf{p}_1, \boldsymbol{\eta}_{x_1})) - \mathbf{f}(t, \mathbf{p}_2, \boldsymbol{\eta}_{x_2}, \mathbf{r}(t, \mathbf{p}_2, \boldsymbol{\eta}_{x_2}))\|, \\ &\leq k_{\mathbf{f}}(t)\|(\mathbf{p}_1, \boldsymbol{\eta}_{x_1}, \mathbf{r}(t, \mathbf{p}_1, \boldsymbol{\eta}_{x_1})) - (\mathbf{p}_2, \boldsymbol{\eta}_{x_2}, \mathbf{r}(t, \mathbf{p}_2, \boldsymbol{\eta}_{x_2}))\|, \\ &\leq k_{\mathbf{f}}(t)(1 + k_{\mathbf{r}})\|(\mathbf{p}_1, \boldsymbol{\eta}_{x_1}) - (\mathbf{p}_2, \boldsymbol{\eta}_{x_2})\|,\end{aligned}$$

for any $t \in [t_0, t_f]$ and any $(\mathbf{p}_1, \boldsymbol{\eta}_{x_1}), (\mathbf{p}_2, \boldsymbol{\eta}_{x_2}) \in \pi_{p,x}(D_\delta; t)$.

By replacing $\mathbf{f}$ by $\bar{\mathbf{f}}$ and D by D_δ, it is valid to apply Theorem 2.4 to

$$\begin{aligned}\dot{\mathbf{u}}(t, \mathbf{p}) &= \bar{\mathbf{f}}(t, \mathbf{u}(t, \mathbf{p})), \\ \mathbf{u}(t_0, \mathbf{p}) &= \mathbf{f}_0(\mathbf{p}),\end{aligned}$$

which admits the unique solution $\mathbf{x}(\cdot, \mathbf{p})$ on $[t_0, t_f]$ for each $\mathbf{p} \in N(\mathbf{p}_0)$. Theorem 2.4 yields that, for each $t \in [t_0, t_f]$, $\mathbf{x}_t \equiv \mathbf{x}(t, \cdot)$ is Lipschitz continuous on a neighborhood $\widetilde{N}(\mathbf{p}_0) \subset N(\mathbf{p}_0)$ of $\mathbf{p}_0$, with Lipschitz constant $k_{\mathbf{x}} \geq 0$. For any $\mathbf{p}_1, \mathbf{p}_2 \in \widetilde{N}(\mathbf{p}_0)$,

$$\begin{aligned}\|\mathbf{y}(t, \mathbf{p}_1) - \mathbf{y}(t, \mathbf{p}_2)\| &= \|\mathbf{r}(t, \mathbf{p}_1, \mathbf{x}(t, \mathbf{p}_1)) - \mathbf{r}(t, \mathbf{p}_2, \mathbf{x}(t, \mathbf{p}_2))\|, \\ &\leq k_{\mathbf{r}}\|(\mathbf{p}_1, \mathbf{x}(t, \mathbf{p}_1)) - (\mathbf{p}_2, \mathbf{x}(t, \mathbf{p}_2))\|, \\ &\leq k_{\mathbf{r}}(1 + k_{\mathbf{x}})\|\mathbf{p}_1 - \mathbf{p}_2\|,\end{aligned}$$

since $\{(t, \mathbf{p}, \mathbf{x}(t, \mathbf{p})) : (t, \mathbf{p}) \in [t_0, t_f] \times \widetilde{N}(\mathbf{p}_0)\} \subset D_\delta$. This demonstrates Lipschitz continuity of $\mathbf{y}_t$ on $\widetilde{N}(\mathbf{p}_0)$, with a Lipschitz constant independent of t. From Theorem 2.4 it also follows that $\mathbf{x}_t$ is L-smooth at $\mathbf{p}_0$, which implies that the mapping $\mathbf{y}_t$ is L-smooth at $\mathbf{p}_0$ for any $t \in [t_0, t_f]$ since $\mathbf{y}_t(\cdot) \equiv \mathbf{r}_t(\cdot, \mathbf{x}_t(\cdot))$ on $N(\mathbf{p}_0)$ and $\mathbf{r}_t$ is L-smooth at $(\mathbf{p}_0, \mathbf{x}_t(\mathbf{p}_0))$.

Define the following mappings:

$$\begin{aligned}&\widetilde{\mathbf{r}}_{(0)} : [t_0, t_f] \times \mathbb{R}^{n_p + n_x} \to \mathbb{R}^{n_y} : (t, \mathbf{d}) \mapsto [\mathbf{r}_t]'(\mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0); \mathbf{d}), \\ &\widetilde{\mathbf{r}}_{(i)} : [t_0, t_f] \times \mathbb{R}^{n_p + n_x} \to \mathbb{R}^{n_y} : (t, \mathbf{d}) \mapsto [\widetilde{\mathbf{r}}_{(i-1),t}]'(\mathbf{m}_{(i)}, [\mathbf{x}_t]^{(i-1)}_{\mathbf{p}_0, \mathbf{M}}(\mathbf{m}_{(i)}); \mathbf{d}), \\ &\qquad \forall i \in \{1, \ldots, k-1\},\end{aligned}$$

which are well-defined since $\mathbf{r}$ is L-smooth on $D_\delta \supset \{(t, \mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0)) : t \in [t_0, t_f]\}$. It will be shown by induction that, for each $i \in \{0, 1, \ldots, k-1\}$, $\widetilde{\mathbf{r}}_{(i)}(\cdot, \mathbf{d})$ is measurable on $[t_0, t_f]$ for each $\mathbf{d} \in \mathbb{R}^{n_p + n_x}$ and $\widetilde{\mathbf{r}}_{(i)}(t, \cdot)$ is Lipschitz continuous on $\mathbb{R}^{n_p + n_x}$ for each $t \in [t_0, t_f]$, with a Lipschitz constant that is independent of t.

Consider the base case and choose any $\mathbf{d} \in \mathbb{R}^{n_p + n_x}$. The first part proceeds as in the proof of Theorem 4.1 in [10]: by construction,

$$\{(t, \mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0)) : t \in [t_0, t_f]\} \subset D_\delta,$$

where $\{(t, \mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0)) : t \in [t_0, t_f]\}$ is compact and D_δ is open. Thus,

$$\{(t, \mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0)) : t \in [t_0, t_f]\} \cap (\mathbb{R}^{1 + n_p + n_x} \setminus D_\delta) = \emptyset.$$

Let $\widetilde{\mathbf{d}} := (0, \mathbf{d})$. There exists $\epsilon > 0$ such that for any $t \in [t_0, t_f]$ and any $\tau \in [0, \epsilon]$,

$$(t, \mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0)) + \tau \widetilde{\mathbf{d}} = (t, (\mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0)) + \tau \mathbf{d}) \in D_\delta;$$

this follows from Lemma 1 in Chapter 2, Section 5 [39]. Since $t \mapsto (\mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0))$ is continuous on $[t_0, t_f]$, the composite mapping $t \mapsto \mathbf{r}(t, (\mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0)) + \tau \mathbf{d})$ is continuous on $[t_0, t_f]$ for each $\tau \in [0, \epsilon]$. The mapping

$$t \mapsto \lim_{\alpha \downarrow 0} \frac{\mathbf{r}(t, (\mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0)) + \alpha \mathbf{d}) - \mathbf{r}(t, \mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0))}{\alpha}$$

is the pointwise limit of a sequence of continuous functions and is therefore measurable on $[t_0, t_f]$, from which it follows that $\widetilde{\mathbf{r}}_{(0)}(\cdot, \mathbf{d})$ is measurable on $[t_0, t_f]$ for each $\mathbf{d} \in \mathbb{R}^{n_p + n_x}$.

Choose any $t \in [t_0, t_f]$. The function $\mathbf{r}_t$ is Lipschitz continuous on $\pi_{p,x}(D_\delta; t)$ and directionally differentiable at $(\mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0))$. By (8), $k_\mathbf{r}$ acts as a Lipschitz constant for $\mathbf{r}_t$ in a neighborhood of $(\mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0))$, and as a result

$$\begin{aligned}\|\widetilde{\mathbf{r}}_{(0)}(t, \mathbf{d}_1) - \widetilde{\mathbf{r}}_{(0)}(t, \mathbf{d}_2)\| &= \|[\mathbf{r}_t]'(\mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0); \mathbf{d}_1) - [\mathbf{r}_t]'(\mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0); \mathbf{d}_2)\|, \\ &\leq k_\mathbf{r}\|\mathbf{d}_1 - \mathbf{d}_2\|, \quad \forall t \in [t_0, t_f], \quad \forall \mathbf{d}_1, \mathbf{d}_2 \in \mathbb{R}^{n_p+n_x},\end{aligned}$$

by Theorem 3.1.2 in [9]. Hence, $\widetilde{\mathbf{r}}_{(0)}(t, \cdot)$ is Lipschitz continuous on $\mathbb{R}^{n_p+n_x}$ for each $t \in [t_0, t_f]$, with Lipschitz constant $k_\mathbf{r}$.

Assume that the claim is true for $i := j \in \{0, 1, \ldots, k-2\}$ and choose any $\mathbf{d} \in \mathbb{R}^{n_p+n_x}$. Corollary 4.2 in [10] implies that $t \mapsto [\mathbf{x}_t]^{(i-1)}_{\mathbf{p}_0,\mathbf{M}}(\mathbf{m}_{(i)})$ is an absolutely continuous mapping on $[t_0, t_f]$ for each $i \in \{1, \ldots, k\}$. Hence, the mapping

$$t \mapsto \begin{bmatrix} \mathbf{m}_{(j+1)} \\ [\mathbf{x}_t]^{(j)}_{\mathbf{p}_0,\mathbf{M}}(\mathbf{m}_{(j+1)}) \end{bmatrix} + \tau\mathbf{d}$$

is absolutely continuous, and therefore measurable, on $[t_0, t_f]$ for any $\tau \geq 0$. By the inductive assumption, the mapping $\widetilde{\mathbf{r}}_{(j)}(\cdot, \boldsymbol{\eta})$ is measurable on $[t_0, t_f]$ for any $\boldsymbol{\eta} \in \mathbb{R}^{n_p+n_x}$ and there exists $k_{\widetilde{\mathbf{r}}_{(j)}} \geq 0$ such that

$$\|\widetilde{\mathbf{r}}_{(j)}(t, \mathbf{d}_1) - \widetilde{\mathbf{r}}_{(j)}(t, \mathbf{d}_2)\| \leq k_{\widetilde{\mathbf{r}}_{(j)}}\|\mathbf{d}_1 - \mathbf{d}_2\|, \quad \forall t \in [t_0, t_f], \quad \forall \mathbf{d}_1, \mathbf{d}_2 \in \mathbb{R}^{n_p+n_x}.$$

Hence,

$$\|\widetilde{\mathbf{r}}_{(j)}(t, \mathbf{d})\| = \|\widetilde{\mathbf{r}}_{(j)}(t, \mathbf{d}) - \widetilde{\mathbf{r}}_{(j)}(t, \mathbf{0}_{n_p+n_x})\| \leq k_{\widetilde{\mathbf{r}}_{(j)}}\|\mathbf{d}\|, \quad \forall (t, \mathbf{d}) \in [t_0, t_f] \times \mathbb{R}^{n_p+n_x}.$$

Consequently, the mapping

$$t \mapsto \frac{\widetilde{\mathbf{r}}_{(j)}(t, (\mathbf{m}_{(j+1)}, [\mathbf{x}_t]^{(j)}_{\mathbf{p}_0,\mathbf{M}}(\mathbf{m}_{(j+1)})) + \tau\mathbf{d}) - \widetilde{\mathbf{r}}_{(j)}(t, \mathbf{m}_{(j+1)}, [\mathbf{x}_t]^{(j)}_{\mathbf{p}_0,\mathbf{M}}(\mathbf{m}_{(j+1)}))}{\tau}$$

is Lebesgue integrable, and therefore measurable, on $[t_0, t_f]$ for any $\tau > 0$ by Lemma 1 in Chapter 1, Section 1 [39]. Then, since $\widetilde{\mathbf{r}}_{(j),t}$ is directionally differentiable at $(\mathbf{m}_{(j+1)}, [\mathbf{x}_t]^{(j)}_{\mathbf{p}_0,\mathbf{M}}(\mathbf{m}_{(j+1)}))$, the mapping

$$t \mapsto \lim_{\alpha \downarrow 0} \frac{\widetilde{\mathbf{r}}_{(j)}(t, (\mathbf{m}_{(j+1)}, [\mathbf{x}_t]^{(j)}_{\mathbf{p}_0,\mathbf{M}}(\mathbf{m}_{(j+1)})) + \alpha\mathbf{d}) - \widetilde{\mathbf{r}}_{(j)}(t, \mathbf{m}_{(j+1)}, [\mathbf{x}_t]^{(j)}_{\mathbf{p}_0,\mathbf{M}}(\mathbf{m}_{(j+1)}))}{\alpha}$$

is well-defined and is measurable on $[t_0, t_f]$ as the pointwise limit of a sequence of measurable functions. Hence, $\widetilde{\mathbf{r}}_{(j+1)}(\cdot, \mathbf{d})$ is measurable on $[t_0, t_f]$ for each $\mathbf{d} \in \mathbb{R}^{n_p+n_x}$.

Again by Theorem 3.1.2 in [9], the finite constant $k_{\widetilde{\mathbf{r}}_{(j)}}$ acts as a Lipschitz constant for $\widetilde{\mathbf{r}}_{(j+1),t} \equiv [\widetilde{\mathbf{r}}_{(j),t}]'(\mathbf{m}_{(j+1)}, [\mathbf{x}_t]^{(j)}_{\mathbf{p}_0,\mathbf{M}}(\mathbf{m}_{(j+1)}); \cdot)$ on $\mathbb{R}^{n_p+n_x}$:

$$\|\widetilde{\mathbf{r}}_{(j+1)}(t, \mathbf{d}_1) - \widetilde{\mathbf{r}}_{(j+1)}(t, \mathbf{d}_2)\| \leq k_{\widetilde{\mathbf{r}}_{(j)}}\|\mathbf{d}_1 - \mathbf{d}_2\|, \quad \forall t \in [t_0, t_f], \quad \forall \mathbf{d}_1, \mathbf{d}_2 \in \mathbb{R}^{n_p+n_x},$$

implying that $\widetilde{\mathbf{r}}_{(j+1)}(t, \cdot)$ is Lipschitz continuous on $\mathbb{R}^{n_p+n_x}$ for each $t \in [t_0, t_f]$, with a Lipschitz constant that is independent of t. The claim is therefore proved by induction.

Define the following mappings:

$$\begin{aligned}&\widetilde{\mathbf{x}}_{(i)} : [t_0, t_f] \to \mathbb{R}^{n_x} : t \mapsto [\mathbf{x}_t]^{(i-1)}_{\mathbf{p}_0,\mathbf{M}}(\mathbf{m}_{(i)}), \quad \forall i \in \{1, \ldots, k\}, \\ &\widetilde{\mathbf{y}}_{(i)} : [t_0, t_f] \to \mathbb{R}^{n_y} : t \mapsto \widetilde{\mathbf{r}}_{(i-1)}(t, \mathbf{m}_{(i)}, [\mathbf{x}_t]^{(i-1)}_{\mathbf{p}_0,\mathbf{M}}(\mathbf{m}_{(i)})), \quad \forall i \in \{1, \ldots, k\}.\end{aligned}$$

Choose any $i := j \in \{1, \ldots, k\}$. For each $\mathbf{d} \in \mathbb{R}^{n_p+n_x}$, the mapping $\widetilde{\mathbf{r}}_{(j-1)}(\cdot, \mathbf{d})$ is measurable on $[t_0, t_f]$. Moreover, there exists $k_{\widetilde{\mathbf{r}}_{(j-1)}} \geq 0$ such that

$$\|\widetilde{\mathbf{r}}_{(j-1)}(t, \mathbf{d}_1) - \widetilde{\mathbf{r}}_{(j-1)}(t, \mathbf{d}_2)\| \leq k_{\widetilde{\mathbf{r}}_{(j-1)}}\|\mathbf{d}_1 - \mathbf{d}_2\|, \quad \forall t \in [t_0, t_f], \quad \forall \mathbf{d}_1, \mathbf{d}_2 \in \mathbb{R}^{n_p+n_x},$$

and

$$\begin{aligned}\|\widetilde{\mathbf{r}}_{(j-1)}(t, \mathbf{d})\| &= \|\widetilde{\mathbf{r}}_{(j-1)}(t, \mathbf{d}) - \widetilde{\mathbf{r}}_{(j-1)}(t, \mathbf{0}_{n_p+n_x})\|, \\ &\leq k_{\widetilde{\mathbf{r}}_{(j-1)}}\|\mathbf{d}\|, \quad \forall (t, \mathbf{d}) \in [t_0, t_f] \times \mathbb{R}^{n_p+n_x}.\end{aligned}$$

It was demonstrated earlier that the mapping $\widetilde{\mathbf{x}}_{(j)} : t \mapsto [\mathbf{x}_t]^{(j-1)}_{\mathbf{p}_0,\mathbf{M}}(\mathbf{m}_{(j)})$ is absolutely continuous on $[t_0, t_f]$. $\widetilde{\mathbf{x}}_{(j)}$ is therefore measurable on $[t_0, t_f]$, from which it follows that $\widetilde{\mathbf{y}}_{(j)}$ is Lebesgue integrable on $[t_0, t_f]$ by Lemma 1 in Chapter 1, Section 1 [39].

Define the following matrix-valued functions:

$$\begin{aligned}\widetilde{\mathbf{X}} &: [t_0, t_f] \to \mathbb{R}^{n_x \times k} : t \mapsto [\widetilde{\mathbf{x}}_{(1)}(t) \quad \cdots \quad \widetilde{\mathbf{x}}_{(k)}(t)],\\ \widetilde{\mathbf{Y}} &: [t_0, t_f] \to \mathbb{R}^{n_y \times k} : t \mapsto [\widetilde{\mathbf{y}}_{(1)}(t) \quad \cdots \quad \widetilde{\mathbf{y}}_{(k)}(t)].\end{aligned}$$

For any $k \in \mathbb{N}$ and any $\mathbf{M} \in \mathbb{R}^{n_p \times k}$, Theorem 2.4 implies that the LD-derivative mapping $t \mapsto [\mathbf{x}_t]'(\mathbf{p}_0; \mathbf{M})$ is the unique solution on $[t_0, t_f]$ of the following ODE system:

$$\begin{aligned}\dot{\mathbf{U}}(t) &= [\bar{\mathbf{f}}_{t\setminus Z_\mathbf{f}}]'(\mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0); (\mathbf{M}, \mathbf{U}(t))),\\ \mathbf{U}(t_0) &= [\mathbf{f}_0]'(\mathbf{p}_0; \mathbf{M}).\end{aligned} \tag{9}$$

By L-smoothness of $\mathbf{q}_t$ and $\mathbf{r}_t$ at $(\mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0))$ for each $t \in [t_0, t_f]$, the LD-derivative chain rule (1) yields

$$\begin{aligned}&[\mathbf{f}_{t\setminus Z_\mathbf{f}} \circ \mathbf{q}_t]'(\mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0); (\mathbf{M}, \mathbf{A}))\\ &= [\mathbf{f}_{t\setminus Z_\mathbf{f}}]'(\mathbf{q}_t(\mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0)); [\mathbf{q}_t]'(\mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0); (\mathbf{M}, \mathbf{A}))),\\ &= [\mathbf{f}_{t\setminus Z_\mathbf{f}}]'(\mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0), \mathbf{r}(t, \mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0)); (\mathbf{M}, \mathbf{A}, [\mathbf{r}_t]'(\mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0); (\mathbf{M}, \mathbf{A})))),\end{aligned}$$

for any $(t, \mathbf{A}) \in [t_0, t_f] \times \mathbb{R}^{n_x \times k}$. Since (9) admits the unique solution $\widetilde{\mathbf{X}}$ on $[t_0, t_f]$,

$$\dot{\widetilde{\mathbf{X}}}(t) = [\mathbf{f}_{t\setminus Z_\mathbf{f}}]' \left(\begin{bmatrix} \mathbf{p}_0 \\ \mathbf{x}(t, \mathbf{p}_0) \\ \mathbf{r}(t, \mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0)) \end{bmatrix}; \begin{bmatrix} \mathbf{M} \\ \widetilde{\mathbf{X}}(t) \\ [\mathbf{r}_t]'(\mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0); (\mathbf{M}, \widetilde{\mathbf{X}}(t))) \end{bmatrix} \right), \tag{10}$$

for almost every $t \in [t_0, t_f]$ and

$$\widetilde{\mathbf{X}}(t_0) = [\mathbf{f}_0]'(\mathbf{p}_0; \mathbf{M}). \tag{11}$$

For each $t \in [t_0, t_f]$ and each $i \in \{1, \ldots, k\}$,

$$\begin{aligned}\widetilde{\mathbf{y}}_{(i)}(t) &= \widetilde{\mathbf{r}}_{(i-1)}(t, \mathbf{m}_{(i)}, [\mathbf{x}_t]^{(i-1)}_{\mathbf{p}_0,\mathbf{M}}(\mathbf{m}_{(i)})),\\ &= [\mathbf{r}_t]^{(i-1)}_{(\mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0)), (\mathbf{M}, [\mathbf{x}_t]'(\mathbf{p}_0; \mathbf{M}))}(\mathbf{m}_{(i)}, [\mathbf{x}_t]^{(i-1)}_{\mathbf{p}_0,\mathbf{M}}(\mathbf{m}_{(i)})),\end{aligned}$$

from which it follows that

$$\begin{aligned}\widetilde{\mathbf{Y}}(t) &= [\mathbf{r}_t]'(\mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0); (\mathbf{M}, [\mathbf{x}_t]'(\mathbf{p}_0; \mathbf{M})),\\ &= [\mathbf{r}_t]'(\mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0); (\mathbf{M}, \widetilde{\mathbf{X}}(t)), \quad \forall t \in [t_0, t_f].\end{aligned} \tag{12}$$

Equation (4) and Lemma 2.9 imply that, for each $t \in [t_0, t_f]$,

$$\mathbf{N} := \mathbf{r}'(t, \mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0); (\mathbf{0}_{1\times k}, \mathbf{M}, [\mathbf{x}_t]'(\mathbf{p}_0; \mathbf{M}))) = [\mathbf{r}_t]'(\mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0); (\mathbf{M}, \widetilde{\mathbf{X}}(t)))$$

is the unique solution of

$$\begin{aligned}\mathbf{0}_{n_y\times k} &= \mathbf{g}'(t, \mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0), \mathbf{y}(t, \mathbf{p}_0); (\mathbf{0}_{1\times k}, \mathbf{M}, [\mathbf{x}_t]'(\mathbf{p}_0; \mathbf{M}), \mathbf{N})),\\ &= [\mathbf{g}_t]'(\mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0), \mathbf{y}(t, \mathbf{p}_0); (\mathbf{M}, \widetilde{\mathbf{X}}(t), \mathbf{N})).\end{aligned}$$

Hence,

$$\mathbf{0}_{n_y\times k} = [\mathbf{g}_t]'(\mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0), \mathbf{y}(t, \mathbf{p}_0); (\mathbf{M}, \widetilde{\mathbf{X}}(t), \widetilde{\mathbf{Y}}(t))), \quad \forall t \in [t_0, t_f]. \tag{13}$$

For each $t \in [t_0, t_f]$, the L-smoothness of $\mathbf{y}_t$ was established earlier; the LD-derivative chain rule yields

$$[\mathbf{y}_t]'(\mathbf{p}_0; \mathbf{M}) = [\mathbf{r}_t]'(\mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0); (\mathbf{M}, [\mathbf{x}_t]'(\mathbf{p}_0; \mathbf{M}))) = \widetilde{\mathbf{Y}}(t), \quad \forall t \in [t_0, t_f].$$

Evaluation of (13) at $t = t_0$ yields the fact that $\widetilde{\mathbf{Y}}(t_0)$ is the unique solution $\mathbf{Y}_0 \in \mathbb{R}^{n_y \times k}$ of the equation system

$$\mathbf{0}_{n_y\times k} = [\mathbf{g}_{t_0}]'(\mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0; (\mathbf{M}, [\mathbf{f}_0]'(\mathbf{p}_0; \mathbf{M}), \mathbf{Y}_0)),$$

since $\mathbf{x}(t_0, \mathbf{p}_0) = \mathbf{x}_0$, $\mathbf{y}(t_0, \mathbf{p}_0) = \mathbf{y}_0$, and $\widetilde{\mathbf{X}}(t_0) = [\mathbf{f}_0]'(\mathbf{p}_0; \mathbf{M})$. The conclusion of the theorem holds by virtue of Equations (10) to (13), and the observation that $\mathbf{y}(t, \mathbf{p}_0) = \mathbf{r}(t, \mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0))$, $\widetilde{\mathbf{X}}(t) = [\mathbf{x}_t]'(\mathbf{p}_0, \mathbf{M})$, and $\widetilde{\mathbf{Y}}(t) = [\mathbf{y}_t]'(\mathbf{p}_0, \mathbf{M})$ for all $t \in [t_0, t_f]$. □

Remark 3.8. If $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{f}_0$ are C^1 on their respective domains, then $Z_{\mathbf{f}} = \emptyset$ and, as expected, (6) simplifies to

$$\begin{aligned}
\dot{\mathbf{X}}(t) &= \frac{\partial \mathbf{f}}{\partial \mathbf{p}}\mathbf{M} + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\mathbf{X}(t) + \frac{\partial \mathbf{f}}{\partial \mathbf{y}}\mathbf{Y}(t),\\
\mathbf{0}_{n_y\times k} &= \frac{\partial \mathbf{g}}{\partial \mathbf{p}}\mathbf{M} + \frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{X}(t) + \frac{\partial \mathbf{g}}{\partial \mathbf{y}}\mathbf{Y}(t),\\
\mathbf{X}(t_0) &= \mathbf{J}\mathbf{f}_0(\mathbf{p}_0)\mathbf{M},
\end{aligned}$$

where the partial derivatives of $\mathbf{f}$ and $\mathbf{g}$ are evaluated at $(t, \mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0), \mathbf{y}(t, \mathbf{p}_0))$, which has been omitted for brevity.

Remark 3.9. Given a regular solution $\mathbf{z}$ of (5) on $[t_0, t_f]\times\{\mathbf{p}_0\}$ through $\{(t_0, \mathbf{p}_0, \mathbf{x}_0, \mathbf{y}_0)\}$ and any nonsingular $\mathbf{M} \in \mathbb{R}^{n_p\times n_p}$, $(\mathbf{X}(t_f), \mathbf{Y}(t_f)) := [\mathbf{z}_{t_f}]'(\mathbf{p}_0; \mathbf{M})$ can be obtained by evaluating the unique solution of the auxiliary nonsmooth DAE system (6) at $t = t_f$. As an element of the lexicographic subdifferential,

$$\begin{bmatrix}\mathbf{J}_{\mathrm{L}}\mathbf{x}_{t_f}(\mathbf{p}_0;\mathbf{M})\\ \mathbf{J}_{\mathrm{L}}\mathbf{y}_{t_f}(\mathbf{p}_0;\mathbf{M})\end{bmatrix} = \mathbf{J}_{\mathrm{L}}\mathbf{z}_{t_f}(\mathbf{p}_0;\mathbf{M})$$

is a computationally relevant object related to the parametric sensitivities of the differential variables $\mathbf{x}$ and algebraic variables $\mathbf{y}$, respectively, at $t = t_f$. It can be furnished by solving the following linear equation system:

$$\begin{bmatrix}\mathbf{X}(t_f)\\ \mathbf{Y}(t_f)\end{bmatrix} = \begin{bmatrix}\mathbf{J}_{\mathrm{L}}\mathbf{x}_{t_f}(\mathbf{p}_0;\mathbf{M})\\ \mathbf{J}_{\mathrm{L}}\mathbf{y}_{t_f}(\mathbf{p}_0;\mathbf{M})\end{bmatrix}\mathbf{M}.$$

Remark 3.10. Mirroring the discussion in Remark 2.5, the right-hand side function

$$(t, \mathbf{A}) \mapsto [\bar{\mathbf{f}}_{t\setminus Z_{\mathbf{f}}}]'(\mathbf{p}_0, \mathbf{x}(t, \mathbf{p}_0); (\mathbf{M}, \mathbf{A}))$$

in (9) need not satisfy the Carathéodory conditions, but the k columns of the matrix-valued function $t \mapsto [\mathbf{x}_t]'(\mathbf{p}_0; \mathbf{M})$ are nonetheless absolutely continuous on $[t_0, t_f]$. However, the k columns of the matrix-valued function $t \mapsto [\mathbf{y}_t]'(\mathbf{p}_0; \mathbf{M})$ are Lebesgue integrable vector-valued functions mapping $[t_0, t_f]$ to $\mathbb{R}^{n_y}$, and therefore may exhibit discontinuities with respect to the independent variable. This observation is also applicable to Corollary 3.11, as illustrated in Example 4.2.

Sensitivities of solutions of (5) with respect to initial data are easily computed by Theorem 3.7.

Corollary 3.11. Suppose that $\mathbf{f} : D_t \times D_x \times D_y \to \mathbb{R}^{n_x}$ and $\mathbf{g} : D_t \times D_x \times D_y \to \mathbb{R}^{n_y}$ satisfy analogous conditions to the hypotheses of Assumption 3.1. Suppose that there exists a regular solution $\mathbf{z}$ of

$$\begin{aligned}
\dot{\mathbf{x}}(t, \mathbf{c}) &= \mathbf{f}(t, \mathbf{x}(t, \mathbf{c}), \mathbf{y}(t, \mathbf{c})),\\
\mathbf{0} &= \mathbf{g}(t, \mathbf{x}(t, \mathbf{c}), \mathbf{y}(t, \mathbf{c})),\\
\mathbf{x}(t_0, \mathbf{c}) &= \mathbf{c},
\end{aligned} \tag{14}$$

on $[t_0, t_f] \times \{\mathbf{c}_0\}$ through $\{(t_0, \mathbf{c}_0, \mathbf{y}_0)\}$ for some $(\mathbf{c}_0, \mathbf{y}_0) \in D_x \times D_y$. Then, for each $t \in [t_0, t_f]$, the mapping $\mathbf{z}_t \equiv \mathbf{z}(t, \cdot)$ is L-smooth at $\mathbf{c}_0$; for any $k \in \mathbb{N}$ and any $\mathbf{M} \in \mathbb{R}^{n_x\times k}$, the LD-derivative mapping

$$\widetilde{\mathbf{Z}} \equiv (\widetilde{\mathbf{X}}, \widetilde{\mathbf{Y}}) : [t_0, t_f] \to \mathbb{R}^{(n_x+n_y)\times k} : t \mapsto [\mathbf{z}_t]'(\mathbf{c}_0; \mathbf{M})$$

is the unique solution (in the sense of Theorem 3.7) of the following DAE system:

$$\begin{aligned}
\dot{\mathbf{X}}(t) &= [\mathbf{f}_{t\setminus Z_{\mathbf{f}}}]'(\mathbf{x}(t,\mathbf{c}_0),\mathbf{y}(t,\mathbf{c}_0);(\mathbf{X}(t),\mathbf{Y}(t))),\\
\mathbf{0}_{n_y\times k} &= [\mathbf{g}_t]'(\mathbf{x}(t,\mathbf{c}_0),\mathbf{y}(t,\mathbf{c}_0);(\mathbf{X}(t),\mathbf{Y}(t))),\\
\mathbf{X}(t_0) &= \mathbf{M}.
\end{aligned} \tag{15}$$

on $[t_0,t_f]$ through $\{(t_0,\mathbf{M},\mathbf{Y}_0)\}$, where $\mathbf{Y}_0\in\mathbb{R}^{n_y\times k}$ is the unique solution of the equation system

$$\mathbf{0}_{n_y\times k} = [\mathbf{g}_{t_0}]'(\mathbf{x}_0,\mathbf{y}_0;(\mathbf{M},\mathbf{Y}_0)).$$

4. Examples

In this section, examples are provided to highlight the theory.

Example 4.1. Consider the following parametric nonsmooth DAEs:

$$\begin{aligned}
\dot{x}(t,p) &= 0.5\,\mathrm{sign}(1-t)\max\{0,p\}y(t,p),\\
0 &= |x(t,p)|+|y(t,p)|-1,\\
x(0,p) &= \arctan(p),
\end{aligned} \tag{16}$$

where $\mathrm{sign}(\cdot)$ is defined as follows:

$$\mathrm{sign}:\mathbb{R}\to\{-1,0,1\}:t\mapsto\begin{cases}1, & \text{if } t>0,\\ 0, & \text{if } t=0,\\ -1, & \text{if } t<0.\end{cases}$$

Let $p_0:=0$, $N(p_0):=(-0.5,0.5)$, $x_0:=0$, and $y_0:=1$. There exists a unique solution $\mathbf{z}\equiv(x,y)$ of (16) on $[0,2]\times N(p_0)$ through

$$\Omega_0 := \{(t,p,\eta_x,\eta_y): t=0, p\in N(p_0), \eta_x=\arctan(p), \eta_y = 1-|\arctan(p)|\}$$

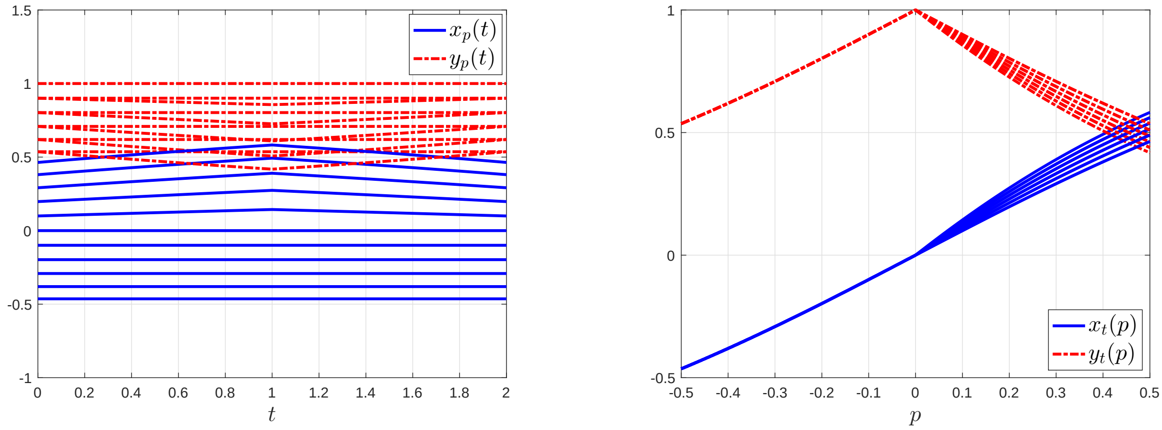
which is given by

$$\mathbf{z}:(t,p)\mapsto\begin{cases}\begin{bmatrix}(\arctan(p)-1)\exp(-0.5pt)+1\\ (1-\arctan(p))\exp(-0.5pt)\end{bmatrix}, & \text{if } (t,p)\in[0,1)\times(0,0.5),\\ \begin{bmatrix}(\beta(p)-1)\exp(0.5p(t-1))+1\\ (1-\beta(p))\exp(0.5p(t-1))\end{bmatrix}, & \text{if } (t,p)\in[1,2]\times(0,0.5),\\ \begin{bmatrix}\arctan(p)\\ 1+\arctan(p)\end{bmatrix}, & \text{if } (t,p)\in[0,2]\times(-0.5,0],\end{cases}$$

where $\beta:(0,0.5)\to(0,1):p\mapsto(\arctan(p)-1)\exp(-0.5p)+1$. See Figure 1 for an illustration.

The solution is regular as $\pi_4\partial g(t,p,x(t,p),y(t,p))=\{1\}$ for all $(t,p)\in[0,2]\times N(p_0)$, since $y(t,p)>0$ for all $(t,p)\in[0,2]\times N(p_0)$. Note that $\mathbf{z}(t,0)=(x(t,0),y(t,0))=(0,1)$ for all $t\in[0,2]$. For any $\mathbf{d}:=(d_1,d_2,d_3)\in\mathbb{R}^3$, $[f_0]'(0;d_1)=d_1$,

$$\begin{aligned}
[f_{t\setminus\{1\}}]'(0,\mathbf{z}(t,0);\mathbf{d}) &= \begin{cases}\lim_{\alpha\downarrow 0}\alpha^{-1}(0.5\max\{0,\alpha d_1\}(1+\alpha d_3)), & \text{if } t\in[0,1),\\ 0, & \text{if } t=1,\\ \lim_{\alpha\downarrow 0}\alpha^{-1}(-0.5\max\{0,\alpha d_1\}(1+\alpha d_3)), & \text{if } t\in(1,2],\end{cases}\\
&= \begin{cases}0.5\max\{0,d_1\}, & \text{if } t\in[0,1),\\ 0, & \text{if } t=1,\\ -0.5\max\{0,d_1\}, & \text{if } t\in(1,2],\end{cases}\\
[g_t]'(0,\mathbf{z}(t,0);\mathbf{d}) &= \lim_{\alpha\downarrow 0}\alpha^{-1}(|\alpha d_2|+|1+\alpha d_3|-1) = |d_2|+d_3.
\end{aligned}$$



(a) $\mathbf{z}(t,p)$ vs. t for various values of $-0.5 < p < 0.5$.

(b) $\mathbf{z}(t,p)$ vs. p for various values of $0 \leq t \leq 2$.

Figure 1: Graphs of solution of (16).

By Theorem 3.7, $\mathbf{z}_t \equiv \mathbf{z}(t,\cdot)$ is L-smooth at $\mathbf{p}_0$ for each $t \in [t_0, t_f]$; for any $m \in \mathbb{R}$, the LD-derivative mapping $\widetilde{\mathbf{Z}} \equiv (\widetilde{X}, \widetilde{Y}) : t \mapsto [\mathbf{z}_t]'(0; m)$ is the unique solution (in the sense of Theorem 3.7) on $[0,2]$ of the following DAE system:

$$\begin{aligned} \dot{X}(t) &= 0.5\,\mathrm{sign}(1-t)\max\{0, m\}, \\ 0 &= |X(t)| + Y(t), \\ X(0) &= m. \end{aligned} \tag{17}$$

The solution of (17) on $[0,2]$ through $\{(0, m, -|m|)\}$ is given by

$$(\widetilde{X}(t), \widetilde{Y}(t)) = \begin{cases} (0.5mt + m, -0.5mt - m) & \text{if } t \in [0,1), m > 0, \\ (-0.5mt + 2m, 0.5mt - 2m) & \text{if } t \in [1,2], m > 0, \\ (m, m) & \text{if } t \in [0,2], m \leq 0. \end{cases} \tag{18}$$

Observe that the initial condition $Y(0)$ in (17) is uniquely determined from $X(0)$ (unlike in (16)), in accordance with Theorem 3.7. See Figure 4a for an illustration; $m = 0$ is admissible.

For any $m \neq 0$, post-multiplying the unique solution $(\widetilde{X}(t), \widetilde{Y}(t))$ of (17) by m^{-1} yields:

$$\mathbf{J}_{\mathrm{L}}\mathbf{z}_t(0; m) = \begin{cases} \{(0.5t + 1, -0.5t - 1)\}, & \text{if } t \in [0,1), m > 0, \\ \{(-0.5t + 2, 0.5t - 2)\}, & \text{if } t \in [1,2], m > 0, \\ \{(1,1)\}, & \text{if } t \in [0,2], m < 0, \end{cases}$$

so that

$$\partial_{\mathrm{L}}\mathbf{z}_t(0) = \begin{cases} \{(0.5t + 1, -0.5t - 1), (1,1)\}, & \text{if } t \in [0,1), \\ \{(-0.5t + 2, 0.5t - 2), (1,1)\}, & \text{if } t \in [1,2]. \end{cases} \tag{19}$$

From the analytic solution, for each $t \in [0,1)$,

$$\mathbf{J}\mathbf{z}_t(p) = \begin{cases} \begin{bmatrix} ((1+p^2)^{-1} - 0.5t(\arctan(p) - 1))\exp(-0.5pt) \\ (-(1+p^2)^{-1} - 0.5t(1 - \arctan(p)))\exp(-0.5pt) \end{bmatrix}, & \text{if } p > 0, \\ \begin{bmatrix} (1+p^2)^{-1} \\ (1+p^2)^{-1} \end{bmatrix}, & \text{if } p < 0, \end{cases}$$

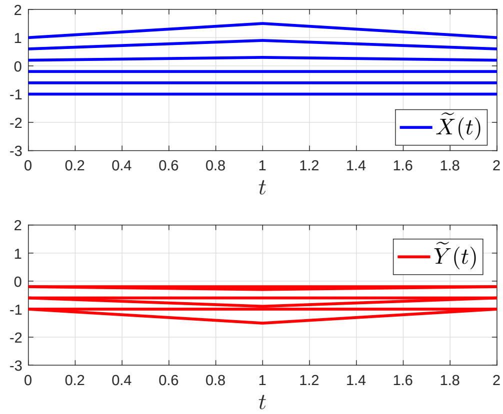
and, for each $t \in [1,2]$,

$$\mathbf{J}\mathbf{z}_t(p) = \begin{cases} \begin{bmatrix} (\beta'(p) + 0.5(t-1)(\beta(p)-1))\exp(0.5p(t-1)) \\ (-\beta'(p) + 0.5(t-1)(1-\beta(p)))\exp(0.5p(t-1)) \end{bmatrix}, & \text{if } p > 0, \\ \begin{bmatrix} (1+p^2)^{-1} \\ (1+p^2)^{-1} \end{bmatrix}, & \text{if } p < 0. \end{cases}$$

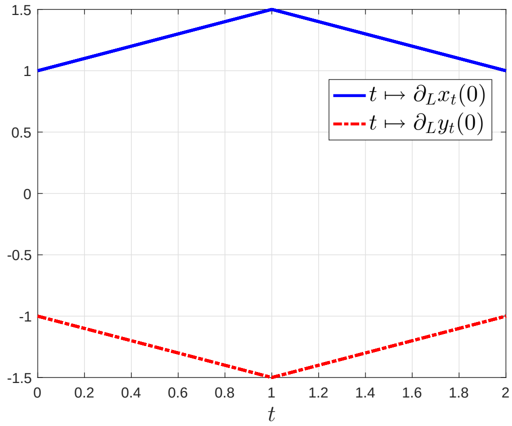
Observe that, for each $t \in [0,2]$,

$$\partial_{\mathrm{B}}\mathbf{z}_t(0) = \begin{cases} \{(1+0.5t, -1-0.5t), (1,1)\}, & \text{if } t \in [0,1), \\ \{(1.5-0.5(t-1), -1.5+0.5(t-1)), (1,1)\}, & \text{if } t \in [1,2]. \end{cases}$$

Noting that $\mathbf{z}_t$ is PC^1 on $N(p_0)$ for each $t \in [0,2]$, it is true that $\partial_{\mathrm{L}}\mathbf{z}_t(0) \subset \partial_{\mathrm{B}}\mathbf{z}_t(0)$ for each $t \in [0,2]$, as expected. See Figure 4b for an illustration.



(a) $\tilde{\mathbf{Z}}(t)$ vs. t for various values of $-1 \leq m \leq 1$.



(b) first element of $\partial_{\mathrm{L}}x_t(0)$ vs. t and $\partial_{\mathrm{L}}y_t(0)$ vs. t.

Figure 2: Graphs of (18) and (19), respectively.

Example 4.2. Consider the following IVP in DAEs:

$$\begin{aligned} \dot{x}_1(t,\mathbf{c}) &= 1 - y(t,\mathbf{c}), \\ \dot{x}_2(t,\mathbf{c}) &= x_2(t,\mathbf{c}), \\ 0 &= \max\{x_1(t,\mathbf{c}), x_2(t,\mathbf{c})\} + |y(t,\mathbf{c})| - 1, \\ x_1(0,\mathbf{c}) &= c_1, \\ x_2(0,\mathbf{c}) &= c_2. \end{aligned} \tag{20}$$

Let $\mathbf{c}_0 := (0,0)$, $y_0 := 1$, and $[t_0, t_f] := [0,1]$. Consider the parameter set

$$C := \{(c_1,c_2) \in \mathbb{R}^2 : 0 \leq c_1 < c_2 \leq 0.3\} \cup \{\mathbf{c}_0\}.$$

The unique solution $\mathbf{z} \equiv (\mathbf{x}, y)$ of (20) on $[0,1] \times [-0.3, 0.3]^2$ through

$$\Omega_0 := \{(t, \eta_{x_1}, \eta_{x_2}, \eta_y) : t = 0, (\eta_{x_1}, \eta_{x_2}) \in C, \eta_y = 1 - \max\{\eta_{x_1}, \eta_{x_2}\}\}$$

is given by

$$\mathbf{z}:(t,\mathbf{c})\mapsto\begin{cases}\begin{bmatrix}c_1+c_2(1-\exp(-t))\\ c_2\exp(-t)\\ 1-c_2\exp(-t)\end{bmatrix}, & \text{if } t\in[0,\tau(\mathbf{c})],\\ \begin{bmatrix}(c_1+c_2(1-\exp(-\tau(\mathbf{c}))))\exp(t-\tau(\mathbf{c}))\\ c_2\exp(-t)\\ 1-(c_1+c_2(1-\exp(-\tau(\mathbf{c}))))\exp(t-\tau(\mathbf{c}))\end{bmatrix}, & \text{if } t\in(\tau(\mathbf{c}),1],\end{cases}$$

where

$$\tau:C\to[0,0.7):(c_1,c_2)\mapsto\begin{cases}\ln\left(\frac{2c_2}{c_1+c_2}\right), & \text{if } (c_1,c_2)\in C\setminus\{\mathbf{c}_0\},\\ 0, & \text{if } (c_1,c_2)=\mathbf{c}_0.\end{cases}$$

See Figure 3 for an illustration. The solution mapping $\mathbf{z}$ is regular; $y(t,\mathbf{c})>0$ for all $(t,\mathbf{c})\in[0,1]\times C$ implies that $\pi_3\partial\mathbf{g}(t,\mathbf{x}(t,\mathbf{c}),y(t,\mathbf{c}))=\{1\}$ for all $(t,\mathbf{c})\in[0,1]\times C$. In fact, there is a unique regular solution of (20) on $[0,1]\times[-0.3,0.3]^2$ through a superset of Ω_0, which can be calculated by inspection and is PC^1 on its domain. However, its complete analytic expression is omitted here to make this example less cumbersome.

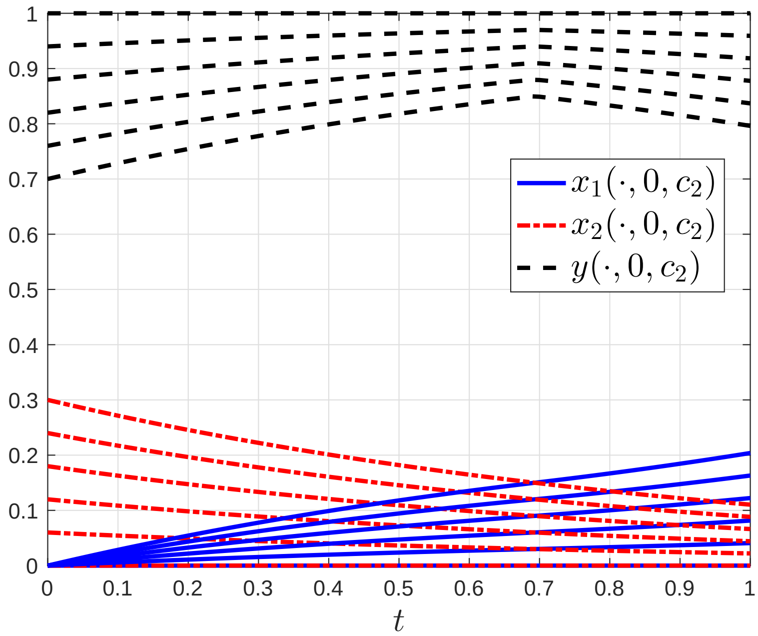


Figure 3: Graphs of solution of (20); $\mathbf{z}(t,\mathbf{c})$ vs. t for $c_1:=0$ and various values of $0\le c_2\le 0.3$.

The right-hand side functions $\mathbf{f}$ and g in (20) are C^1 and PC^1 on $\mathbb{R}^3$, respectively. Note that $\mathbf{z}(t,\mathbf{0}_2)=(\mathbf{x}(t,\mathbf{0}_2),y(t,\mathbf{0}_2))=(0,0,1)$ for all $t\in[0,1]$. Let

$$\mathbf{A}:=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\\ a_{31} & a_{32}\end{bmatrix}\in\mathbb{R}^{3\times 2}.$$

For any $t \in [0,1]$ and any $\mathbf{d} := (d_1, d_2, d_3) \in \mathbb{R}^3$,

$$\begin{aligned}
[g_t]^{(0)}_{\mathbf{z}(t,\mathbf{0}_2),\mathbf{A}}(\mathbf{d}) &= [g_t]'(0,0,1;\mathbf{d}),\\
&= \lim_{\alpha\downarrow 0} \alpha^{-1}(\max\{\alpha d_1, \alpha d_2\} + |1+\alpha d_3| - 1),\\
&= \max\{d_1, d_2\} + d_3,\\
[g_t]^{(1)}_{\mathbf{z}(t,\mathbf{0}_2),\mathbf{A}}(\mathbf{d}) &= [[g_t]^{(0)}_{\mathbf{z}(t,\mathbf{0}_2),\mathbf{A}}]'(a_{11}, a_{21}, a_{31};\mathbf{d}),\\
&= \lim_{\alpha\downarrow 0} \alpha^{-1}(\max\{a_{11}+\alpha d_1, a_{21}+\alpha d_2\} - \max\{a_{11}, a_{21}\} + \alpha d_3),\\
&= \begin{cases} d_1 + d_3, & \text{if } a_{11} > a_{21} \text{ or } a_{11} = a_{21} \text{ and } d_1 \geq d_2,\\ d_2 + d_3, & \text{if } a_{11} < a_{21} \text{ or } a_{11} = a_{21} \text{ and } d_1 < d_2. \end{cases}
\end{aligned}$$

Therefore, for any $t \in [0,1]$,

$$\begin{aligned}
[\mathbf{f}_t]'(\mathbf{z}(t,\mathbf{0}_2);\mathbf{A}) &= \mathbf{J}\mathbf{f}_t(\mathbf{z}(t,\mathbf{0}_2))\mathbf{A} = \begin{bmatrix} -a_{31} & -a_{32}\\ -a_{21} & -a_{22}\end{bmatrix},\\
[g_t]'(\mathbf{z}(t,\mathbf{0}_2);\mathbf{A}) &= \begin{cases} \begin{bmatrix} a_{11}+a_{31} & a_{12}+a_{32}\end{bmatrix}, & \text{if } a_{11} > a_{21} \text{ or } a_{11} = a_{21} \text{ and } a_{12} \geq a_{22},\\ \begin{bmatrix} a_{21}+a_{31} & a_{22}+a_{32}\end{bmatrix}, & \text{if } a_{11} < a_{21} \text{ or } a_{11} = a_{21} \text{ and } a_{12} < a_{22}. \end{cases}
\end{aligned}$$

Corollary 3.11 can be applied as follows: choose any directions matrix

$$\mathbf{M} := \begin{bmatrix} m_{11} & m_{12}\\ m_{21} & m_{22}\end{bmatrix} \in \mathbb{R}^{2\times 2},$$

satisfying

$$0 < m_{22} < m_{11} < m_{21} < m_{12} \leq 0.3$$

(which guarantees its nonsingularity). Consider the following auxiliary nonsmooth DAE system:

$$\begin{aligned}
\dot{\mathbf{X}}(t) &= \begin{bmatrix} -Y_1(t) & -Y_2(t)\\ -X_{21}(t) & -X_{22}(t)\end{bmatrix},\\
\mathbf{Y}(t) &= \begin{cases} \begin{bmatrix} -X_{11}(t) & -X_{12}(t)\end{bmatrix}, & \text{if } X_{11}(t) > X_{21}(t) \text{ or } X_{11}(t) = X_{21}(t) \text{ and } X_{12}(t) \geq X_{22}(t),\\ \begin{bmatrix} -X_{21}(t) & -X_{22}(t)\end{bmatrix}, & \text{if } X_{11}(t) < X_{21}(t) \text{ or } X_{11}(t) = X_{21}(t) \text{ and } X_{12}(t) < X_{22}(t),\end{cases}\\
\mathbf{X}(0) &= \mathbf{M},
\end{aligned} \tag{21}$$

which admits the unique solution (in the sense of Corollary 3.11) $\widetilde{\mathbf{Z}} \equiv (\widetilde{\mathbf{X}}, \widetilde{\mathbf{Y}}) : t \mapsto [\mathbf{z}_t]'(\mathbf{c}_0;\mathbf{M})$ on $[0,1]$ through $\{(0, \mathbf{M}, [-m_{21} \quad -m_{22}])\}$ given by

$$\widetilde{\mathbf{Z}} : t \mapsto \begin{bmatrix} m_{11} + m_{21}(1-\exp(-t)) & m_{12} + m_{22}(1-\exp(-t))\\ m_{21}\exp(-t) & m_{22}\exp(-t)\\ -m_{21}\exp(-t) & -m_{22}\exp(-t)\end{bmatrix},$$

if $t \in [0, \tau(\mathbf{m}_{(1)})]$, and

$$\widetilde{\mathbf{Z}} : t \mapsto \begin{bmatrix} \beta(\mathbf{m}_{(1)})\exp(t-\tau(\mathbf{m}_{(1)})) & \gamma(\mathbf{m}_{(1)}, \mathbf{m}_{(2)})\exp(t-\tau(\mathbf{m}_{(1)}))\\ m_{21}\exp(-t) & m_{22}\exp(-t)\\ -\beta(\mathbf{m}_{(1)})\exp(t-\tau(\mathbf{m}_{(1)})) & -\gamma(\mathbf{m}_{(1)}, \mathbf{m}_{(2)})\exp(t-\tau(\mathbf{m}_{(1)}))\end{bmatrix},$$

if $t \in (\tau(\mathbf{m}_{(1)}), 1]$, where

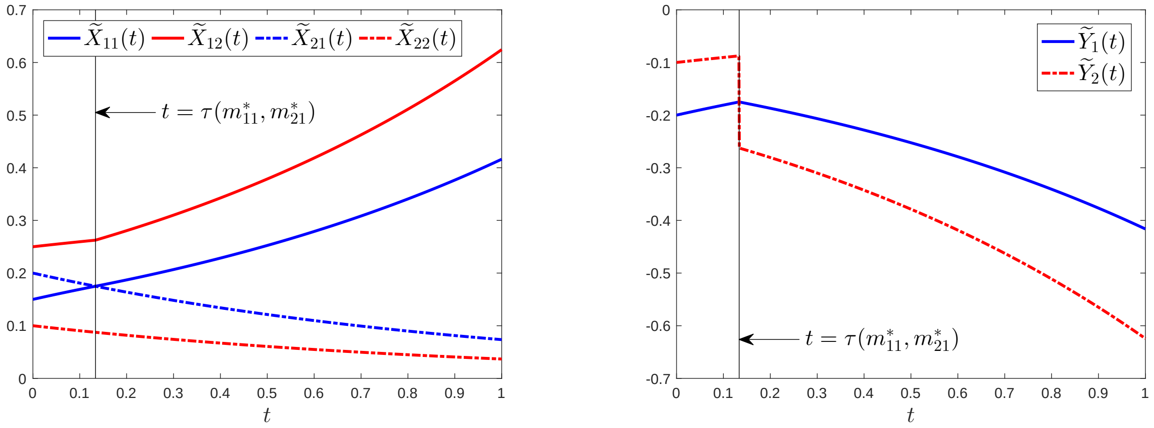
$$\begin{aligned}
\beta &: \mathbf{m}_{(1)} \mapsto m_{11} + m_{21}(1-\exp(-\tau(\mathbf{m}_{(1)}))),\\
\gamma &: (\mathbf{m}_{(1)}, \mathbf{m}_{(2)}) \mapsto m_{12} + m_{22}(1-\exp(-\tau(\mathbf{m}_{(1)}))).
\end{aligned}$$

The mappings $\widetilde{\mathbf{X}}$ and $\widetilde{Y}_1$ are absolutely continuous on $[0,1]$ but

$$\widetilde{Y}_2 : t \mapsto \begin{cases} -m_{22}\exp(-t), & \text{if } t \in [0, \tau(\mathbf{m}_{(1)})], \\ -(m_{12} + m_{22}(1 - \exp(-\tau(\mathbf{m}_{(1)}))))\exp(t - \tau(\mathbf{m}_{(1)})), & \text{if } t \in (\tau(\mathbf{m}_{(1)}), 1], \end{cases}$$

is not continuous at $\tau(\mathbf{m}_{(1)}) \in (0,1)$ since $-m_{22}\exp(-\tau(\mathbf{m}_{(1)})) > -(m_{12} + m_{22}(1 - \exp(-\tau(\mathbf{m}_{(1)}))))$. See Figure 4 for an illustration with directions matrix

$$\mathbf{M}^* := \begin{bmatrix} 0.15 & 0.25 \\ 0.2 & 0.1 \end{bmatrix}.$$



(a) $\tilde{\mathbf{X}}(t)$ vs. t.

(b) $\tilde{\mathbf{Y}}(t)$ vs. t.

Figure 4: Graphs of solution of (21) with directions matrix $\mathbf{M} := \mathbf{M}^*$. Here $\tau(\mathbf{m}^*_{(1)}) = 0.1335$.

Post-multiplying $\widetilde{\mathbf{Z}}(t_f)$ by $\mathbf{M}^{-1}$ furnishes the following L-derivative:

$$\mathbf{J}_{\mathrm{L}}\mathbf{z}_{t_f}(\mathbf{0}_2; \mathbf{M}) = \begin{bmatrix} \exp(1 - \tau(\mathbf{m}_{(1)})) & (1 - \exp(-\tau(\mathbf{m}_{(1)})))\exp(1 - \tau(\mathbf{m}_{(1)})) \\ 0 & \exp(-1) \\ -\exp(1 - \tau(\mathbf{m}_{(1)})) & -(1 - \exp(-\tau(\mathbf{m}_{(1)})))\exp(1 - \tau(\mathbf{m}_{(1)})) \end{bmatrix}.$$

From the analytic solution with $0 < c_1 < c_2 \leq 0.3$,

$$\mathbf{J}\tau(\mathbf{c}) = \begin{bmatrix} -\frac{1}{c_1 + c_2} & \frac{c_1}{c_2(c_1 + c_2)} \end{bmatrix},$$

so that

$$\mathbf{J}\mathbf{z}_{t_f}(\mathbf{c}) = \begin{bmatrix} \exp(1 - \tau(\mathbf{c})) & (1 - \exp(-\tau(\mathbf{c})))\exp(1 - \tau(\mathbf{c})) \\ 0 & \exp(-1) \\ -\exp(1 - \tau(\mathbf{c})) & -(1 - \exp(-\tau(\mathbf{c})))\exp(1 - \tau(\mathbf{c})) \end{bmatrix}.$$

Let $\mathbf{c}_{(j)} := (m_{11}/j, m_{21}/j)$ for each $j \in \mathbb{N}$. Then $\tau(\mathbf{c}_{(j)}) = \tau(\mathbf{m}_{(1)})$ for each $j \in \mathbb{N}$ and $\lim_{j\to\infty} \mathbf{J}\mathbf{z}_{t_f}(\mathbf{c}_{(j)}) = \mathbf{J}_{\mathrm{L}}\mathbf{z}_{t_f}(\mathbf{0}_2; \mathbf{M}) \in \partial_{\mathrm{L}}\mathbf{z}_{t_f}(\mathbf{0}_2) \subset \partial_{\mathrm{B}}\mathbf{z}_{t_f}(\mathbf{0}_2)$, as expected.

5. Conclusions

A theory to compute lexicographic derivatives of solutions of nonsmooth parametric DAEs has been developed. These generalized derivatives are computationally relevant and furnished via the solution of an auxiliary nonsmooth DAE system. The part of this solution mapping that is associated with the algebraic variables exhibits features that are unlike the original nonsmooth parametric DAEs of interest. Namely, it need not be continuous with respect to the independent variable and its initial condition is uniquely determined from the algebraic constraints of the auxiliary nonsmooth DAE system.

Forward sensitivity functions for Carathéodory index-1 semi-explicit DAEs have thus been characterized. Index refers here to a generalized differential index, which is formulated in terms of the projections of Clarke Jacobians being of maximal rank. Existence and regularity of a solution of the nonsmooth parametric DAEs need only be assumed on a finite horizon and at one parameter value for the theory to be applicable. This work is a natural extension of the classical sensitivity results for the analogous smooth case. Numerical solution of large-scale instances of the DAE system (6) will require automatic methods for evaluation of the LD-derivatives appearing in (6), which is facilitated by a recently developed vector forward mode of automatic differentiation for LD-derivative evaluation [7]. Moreover, developing tractable methods for simulating the auxiliary nonsmooth DAE systems found here is an avenue for future work. Other possible directions for future work include extending the results to "high-index" nonsmooth DAEs and adjoint sensitivity results for nonsmooth DAEs.

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